Mathematical Tables
and other
Aids to Computation

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Some Monte Carlo Experiments in Computing Multiple Integrals

1. **Introduction.** Despite the large volume of theoretical material on Monte Carlo methods which has appeared in the last five years, relatively little has been published on the experimental side of this technique [1]. The numerical analyst who is confronted with production computation is frequently anxious to obtain the "feel" of the method by looking at the results of the technique when applied to simple problems. It is towards this end that we present here the results of computing the volumes of n-dimensional hyperspheres ($n = 2, 3, \cdots, 12$) by Monte Carlo methods. These computations were carried out on the National Bureau of Standards Eastern Automatic Computer (SEAC).

2. **The Problem Solved:** An n-dimensional hypersphere H_n is defined by the inequality

$$x_1^2 + x_2^2 + \cdots + x_n^2 \leq 1. \tag{1}$$

Its volume V_n is given by

$$V_n = \frac{2}{n}\frac{\pi^{n/2}}{\Gamma(n/2)}; \tag{2}$$

if E_n designates the hypercube

$$-1 \leq x_i \leq 1 \quad (i = 1, 2, \cdots, n) \tag{3}$$

then the fraction of the volume of E_n which is contained in H_n is

$$v_n = \frac{\pi^{n/2}}{n2^{n-1}\Gamma(n/2)}. \tag{4}$$

The quantity v_n will be called the normalized volume. The ratio v_n may be computed in Monte Carlo style by generating a pseudo-random sequence of points $P_1, P_2, \cdots$ which lie in E_n and finding the success ratio, *i.e.*, the fraction of the points which lie in H_n. In the computations carried out, no use was made of the symmetries of H_n, nor was any attempt made to carry out any of the ideas of "importance sampling" [2].

3. **Generation of Pseudo-Random Sequences:** Two methods were employed to generate a pseudo-random sequence [3]. Method I is that of successive low order multiplication by $5^{17} \cdot 2^{-42}$. This can be described by

$$\begin{aligned} u_{j+1} &= 2^{-42}v_j \\ v_{j+1} &= 5^{17}v_j \pmod{2^{42}}, \quad v_0 = 1. \end{aligned} \tag{5}$$

Method II is one which uses modified Fibonacci sequences. Starting from two

arbitrarily given values $u_0, u_1 \leq 4$, we form successively

$$u_{j+1} = u_j + u_{j-1} \mod 4. \tag{6}$$

The interest in the second method lies in its utilization of addition as opposed to multiplication and raises the possibility of consuming less time. Some preliminary statistical work on these sequences [4] indicated that it was better to select every second number of the sequence (6). With the aim of scrambling the fundamental sequence (6) even further, we introduced skips of length 1, 2, 3 in (6) depending upon whether the last hexidecimal digit of the last number selected was $=1$, 2, or 0, mod 3. In the runs which were tabulated under Method II, the starting values $u_0 = 0$ and $u_1 = 2^{-42}$ were selected. In the runs tabulated under Method II′, the starting values $u_0 = \pi$, $u_1 = 5^{17} \cdot 2^{-42}$ were used.

To generate a sequence of points $P_1, P_2, \cdots$, in E_n, the following scheme was adopted. The fundamental sequence was normalized so that $-1 \leq u_j \leq 1$. The computing machine then inquired: is $u_1^2 + u_2^2 \leq 1$? If so, is $u_1^2 + u_2^2 + u_3^2 \leq 1$?, $\cdots$, etc. If all these questions were answered affirmatively, then P_1: $(u_1, \cdots, u_n)$ and $P_1 \epsilon H_n$. If the question was answered negatively at some point, say $u_1^2 + u_2^2 + \cdots + u_p^2 > 1$, $p \leq n$, then $u_1, \cdots, u_p$ were considered as the first p coordinates of a point P_1 not in H_n. The remaining coordinates $u_{p+1}, \cdots, u_n$ were then not discarded, but the same procedure was repeated for P_2 beginning with u_{p+1}. That is, if $P_1 \epsilon H_n$ then P_2: $(u_{n+1}, u_{n+2}, \cdots, u_{n+p'}, \cdots)$, but if P_1 is not in H_n, then P_2 has its first coordinate beginning with u_{p+1}.

4. **Quality of Results:** According to statistical theory [5], the probable error (error at the 50% level) in a computation with N points is

$$e_{50} = .6745 \sqrt{\frac{v_n(1 - v_n)}{N}}. \tag{7}$$

while the error at the 95% level is given by

$$e_{95} = 1.96 \sqrt{\frac{v_n(1 - v_n)}{N}}. \tag{8}$$

With Method I, the observed error was less than e_{95} in 10 of the twelve cases and exceeded it by about 20% in two of the twelve cases. The quality of Method II is somewhat dubious. The error estimate e_{95} was exceeded in two cases out of six, and in the case $n = 9$, the discrepancy is pronounced. Convergence seems to be taking place towards the wrong value. Method II′ (which differs from Method II only in the starting values) seems to be considerably better and on a par with Method I. In view of the erratic nature of the results of Method II and its apparent sensitivity to starting values, we do not recommend that it be used for purposes of multiple integration. Method I seems to be satisfactory and the results are in conformity with the $1/\sqrt{N}$ law for accuracy. This law is extremely conservative and precludes many computations from being carried out economically, and if, indeed, the present speed of electronic calculators were increased by, say, a factor of 100 this would result in only one more decimal place of accuracy for a fixed amount of computing time. As it is, using Method I, the approximate

TABLE I

Computation of Normalized Volume of n-Dimensional Hypersphere by Sampling

Method I

No. of Points Sampled	$n = 2$	$n = 3$	$n = 4$	$n = 5$
32	0.8750000000	0.5000000000	0.2812500000	0.2187500000
64	.8281250000	.5000000000	.3125000000	.2187500000
128	.7812500000	.5468750000	.3046875000	.2187500000
256	.7929687500	.5625000000	.3242187500	.1835937500
512	.7851562500	.5449218750	.3398437500	.2011718750
1024	.8027343750	.5341796875	.3183593750	.1757812500
2048	.7880859375	.5307617188	.3144531250	.1708984375
4096	.7817382813	.5219726563	.3022460938	.1638183594
8192	.7860107422	.5224609375	.3067626953	.1647949219
16384	.7907714844	.5299682617	.3120117188	.1635742188
32768	.7890014648	.5272216797	.3111267090	.1646423340
Exact Answer	0.785398	0.523599	0.308425	0.164493
Error	0.003603	0.003623	0.002702	0.000149
Predicted Error at 95% Level	0.004445	0.005408	0.005001	0.004014

No. of Points Sampled	$n = 6$	$n = 7$	$n = 8$	$n = 9$
32	0.1250000000	0.0000000000	0.0000000000	0.0000000000
64	.1562500000	.0312500000	.0156250000	.0156250000
128	.1250000000	.0625000000	.0390625000	.0234375000
256	.1054687500	.0468750000	.0273437500	.0156250000
512	.1113281250	.0449218750	.0312500000	.0214843750
1024	.0917968750	.0361328125	.0195312500	.0126953125
2048	.0825195313	.0317382813	.0156250000	.0087890625
4096	.0778808594	.0319824219	.0151367188	.0078125000
8192	.0822753906	.0346679688	.0144042969	.0065917969
16384	.0800170898	.0342407227	.0136108398	.0056762695
32768	.0792236328	.0344543457	.0142517090	.0057983398
65536				.0061340332
Exact Answer	0.080746	0.036912	0.015854	0.006442
Error	0.001522	0.002458	0.001602	0.000308
Predicted Error at 95% Level	0.002950	0.002041	0.001352	0.000612

TABLE I—*Continued*

No. of Points Sampled	$n = 10$	$n = 11$	$n = 12$
32	0.0000000000	0.0000000000	0.0000000000
64	.0000000000	.0000000000	.0000000000
128	.0078125000	.0078125000	.0000000000
256	.0039062500	.0039062500	.0000000000
512	.0078125000	.0039062500	.0019531250
1024	.0048828125	.0019531250	.0009765625
2048	.0024414063	.0009765625	.0004882813
4096	.0024414063	.0009765625	.0002441406
8192	.0021972656	.0007324219	.0001220703
16384	.0015869141	.0006713867	.0002441406
32768	.0017700195	.0007934570	.0003051758
65536	.0022277832	.0010223389	.0004425049
Exact Answer	0.002490	0.000920	0.000326
Error	0.000262	0.000102	0.000117
Predicted Error at 95% Level	0.000540	0.000232	0.000138

Method II

Number of Points Sampled	$n = 2$	$n = 3$	$n = 4$	$n = 5$
16	0.125	0.125	0.0625	0.000
32	.46875	.3125	.1875	.03125
64	.65625	.453125	.28125	.109375
128	.7421875	.5000000	.3046875	.1171875
256	.7695313	.5390625	.3125000	.1406250
512	.7851563	.5214844	.3203125	.1503906
1024	.7783203	.5380859	.3183594	.1572266
2048	.7792969	.5327148	.31591768	.1645507
4096	.7861328		.31494141	.1594238
8192	.7784424		.31323242	
16384	.7832642			
32768	.7859802			
65536	.7859192			
Exact Answer	0.785398	0.5235987	0.308425	0.164493
Error	0.000521	0.0091	0.004807	0.0051
Predicted Error at 95% Level	0.003143	0.021631	0.010001	0.011353

TABLE I—*Continued*

Number of Points Sampled	$n = 6$	$n = 9$
16	0.000	0.000
32	.03125	.000
64	.078125	.000
128	.0859375	.0078125
256	.0742188	.0039063
512	.0761719	.0019531
1024	.0791016	.0029297
2048	.0786133	.0053711
4096	.0764160	.0043945
8192	.0740967	.0039063
16384		.0040894
32768		.0038147
Exact Answer	0.0807455	0.0064424
Error	0.0066	0.0026
Predicted Error at 95% Level	0.005900	0.000866

Method II′

Number of Points Sampled	$n = 2$	$n = 9$
32	0.8125000000	0.0312500000
64	.7500000000	.0156250000
128	.7812500000	.0156250000
256	.7695312500	.0078125000
512	.7714843750	.0039062500
1024	.7822265625	.0048828125
2048	.7812500000	.0048828125
4096	.7800292969	.0053710938
8192	.7800292969	.0072021484
16384		.0068359375
32768		.0073852539
Exact Answer	0.785398	0.006442
Error	0.005369	0.000943
Predicted Error at 95% Level	0.008890	0.000866

amount of time required to compute v_n is $2nN\mu$, where μ is the multiplication time of the machine. The case $n = 12, N = 65{,}536$ consumed about $1\frac{1}{2}$ hours on SEAC.

5. **Some General Observations on Multiple Integration:** One reason for carrying out the experiments described is this. A pseudo-random sequence is one which has passed (in some vague sense) a certain number of statistical tests. It is not clear *a priori* that by combining its consecutive elements into n-tuples we obtain a pseudo-random sequence in n dimensions. Therefore the above experiments constitute additional tests for the randomness of the sequence and, when successful, increase our confidence in its use in multiple integration.

For purposes of integration, it is not necessary to employ pseudo-random sequences; it suffices to employ sequences which are equidistributed [6]. Such sequences can be defined rigorously and the integration of sufficiently smooth functions can be carried out with a theoretical error of order $O(1/N)$ [7].

To obtain a picture of the difference in the speed of convergence, compare the final two tables. In the first of these tables, the integral $\int_0^1 x\,dx$ was computed using the sequence (5). In the second table the same integral was computed using the sequence

$$u_j = \left[j\frac{\sqrt{2}}{2}\right], \quad [x] = \text{fractional part of } x \tag{9}$$

for which the theoretical error estimate is $O(1/N)$. The manner in which the former computation obeys the $1/\sqrt{N}$ law, and the latter the $1/N$ law is striking.

To integrate in n-dimensions, one makes use of the points P_j: $([j\xi_1], [j\xi_2], \cdots, [j\xi_n])$ where ξ_i are n linearly independent irrational numbers belonging to some algebraic number field. The error estimate is again $O(1/N)$, but of course one must contend with a possible increase in the size of the constant which is implicit in the term $O(1/N)$.

Table III presents the results of an integration carried out in four dimensions using $\xi_1 = \frac{\sqrt{2}}{2}$, $\xi_2 = \sqrt{3}$, $\xi_3 = \frac{\sqrt{6}}{3}$, $\xi_4 = \sqrt{10}$. It would appear that these figures satisfy the $1/N$ law although the associated constant is somewhat larger than that of the computation in Table II.

One final observation. The use of the word "Monte Carlo" or "sampling" in connection with quadratures camouflages the following computational fact of life: having agreed to invest in an N point computation, we proceed to form N fixed points $P_1, \cdots, P_N$ and use the approximation

$$\int_V f(P)dv = V/N \sum_{j=1}^{N} f(P_j). \tag{10}$$

This is tantamount to the use of an N point rule with abscissas P_j and weights equal to $1/N$. Such an N point rule is incredibly inaccurate. For any fixed set of N points, we can, in principle at least, select the weights so that the error is minimized in some sense [8]. Or we can select both abscissas and weights so that this error is minimized. It might prove fruitful to look into the possibility of obtaining best quadrature formulas of high order and in spaces of high dimension.

TABLE II

No. of Points Sampled	Computation of $\int_0^1 x\,dx$ Using the Sequence u_j Described by (5)	Computation of $\int_0^1 x\,dx$ Using the Sequence $\left[j\frac{\sqrt{2}}{2}\right]$
2	0.3703920880	0.5606601718
4	.5216710794	.5177669530
8	.5064453536	.5569805153
16	.4547871881	.5104076401
32	.5015472909	.5110118896
64	.5204149952	.4965953886
128	.5268724155	.4990123865
256	.5070362771	.4999401325
512	.5122595896	.4998424994
1024	.5147580545	.4996472331
2048	.5140853905	.4997449819
4096	.5083722340	.4999404795
8192	.5031562483	.5000873340
16384	.5026730074	.5000148320
32768	.5019506662	.5000224160
65536	.5028558372	.5000070665
131072	.5016144215	.4999992555
262144	.5008443891	.4999988924

TABLE III

Computation of $\int_0^1\int_0^1\int_0^1\int_0^1 e^{x_1x_2x_3x_4}dx_1dx_2dx_3dx_4$ Using the Sequence $\left(\left[j\frac{\sqrt{2}}{2}\right],[j\sqrt{3}],\left[j\frac{\sqrt{6}}{3}\right],[j\sqrt{10}]\right)$

No. of Points Sampled	
2	1.0556385354
4	1.0646192182
8	1.0592766300
16	1.0615566540
32	1.0626118843
64	1.0586261203
128	1.0657314237
256	1.0673118827
512	1.0668402647
1024	1.0681499894
2048	1.0685418201
4096	1.0685544568
8192	1.0688021121
16384	1.0691568057
32768	1.0691964247
Exact Answer	1.06939761

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1. See, however: J. TODD, "Experiments in the solution of differential equations by Monte Carlo methods," *J. Wash. Acad. of Sciences*, v. 44, 1954, p. 377–381; N. METROPOLIS, A. W. ROSENBLUTH, M. N. ROSENBLUTH, A. H. TELLER, & E. TELLER, "Equation of state calculations by fast computing machines," *J. Chem. Phys.*, v. 21, 1953, p. 1087–1092; R. D. RICHTMYER, "The evaluation of definite integrals and a quasi-Monte Carlo method based on the properties of algebraic numbers," Los Alamos Rep. 13, Oct. 1951.

2. H. KAHN, "Modification of the Monte Carlo method," Seminar on Scientific Computation, International Business Machines Corp., November 1949.

3. For more detail on these methods see: O. TAUSSKY & J. TODD, "The generation and testing of pseudo-random numbers on SEAC." To appear.

4. *Ibid.*

5. See, *e.g.*, H. KAHN, *loc. cit.*

6. See, *e.g.*, G. PÓLYA & G. SZEGÖ, *Aufgaben und Lehrsätze aus der Analysis*, Dover Publications, Inc., New York, v. 1, 1945, p. 70–77.

7. RICHTMYER, *loc. cit.*

8. ARTHUR SARD, "Best approximate integration formulas," *Am. J. Math.*, v. 71, 1949, p. 80–91; S. M. NIKOLSKII, "K voprosa ob otzenkakh priblizhenii kvadraturnimi formulami," *Uspekhi Math. Nauk*, v. 51, p. 165–177; PHILIP DAVIS, "Errors of numerical approximation for analytic functions," *J. Rat. Mech. and Anal.*, v. 2, 1953, p. 303–313.

Generating and Testing Pseudo Random Numbers on the IBM Type 701

1. Introduction. With the increased use of large-scale digital computing machines many problems too complicated to solve analytically, or even numerically, have been solved by the so-called Monte Carlo method. The Monte Carlo technique employs repeated sampling to evaluate integrals or to simulate physical problems directly and requires a large supply of random numbers distributed in a manner resembling the true distribution of the physical quantities which they represent.

A set of random numbers of any specified distribution can be, and usually is, obtained from a set of random numbers uniformly distributed on the interval 0 to 1. If we are given the density function $f(x)$ of the desired distribution and can integrate this density function in closed form, the most direct way to get the desired set of random numbers $\{N_i\}$ from a set of uniformly distributed random numbers, $\{n_i\}$, is to solve for N_i in the equation

$$n_i = \int_{-\infty}^{N_i} f(x)\,dx. \tag{1}$$

However, the number of cases where the integral can be evaluated and the resulting equation solved explicitly for N_i is small, especially when such solution must be simple enough to be repeated the large number of times required for an extensive table. In some cases the density function $f(x)$ may be approximated by some simpler function, a polynomial say, to afford an approximation of (1) which can easily be solved for N_i.

In some other cases there are special properties of the distribution of N_i by which one is able to get the new set of numbers quite directly from the original set. One example is the normal distribution, or error function, a common distribution of random quantities encountered in physical problems. From the central limit theorem of probability we know that sums of uniformly distributed random variables will approach the normal distribution as the number of random variables

being summed becomes large. In practice, adding the uniformly distributed random numbers in relatively small blocks, on the order of 10, produces a satisfactory set of normally distributed random numbers for many physical applications. In other cases, especially where the extreme values of the distribution are important, such an approximation would not be satisfactory.

When special methods are not available we can always fall back on a sampling method proposed by von NEUMANN [9]. This method is quite general and can be applied to any distribution for which the density function is known, although it is inefficient in some extreme cases.

2. Generating uniformly distributed random numbers. The problem here then is to find a suitable method for generating uniformly distributed random numbers, which we shall refer to simply as random numbers throughout the remainder of this paper. There are tables of random digits available, notably those of TIPPETT [8], KENDALL and SMITH [3], and the RAND Corporation's digits and normal deviates [11]. The loading and storage of such tables within a high speed computer is inefficient. Much to be preferred is a scheme for automatically generating a sequence of random numbers within the machine and to generate them as they are used rather than storing a large table of such numbers. This limits the possibilities to a numerical method, and several schemes have been proposed. Many of these are described by VOTAW and RAFFERTY [10], and TAUSSKY and TODD [7].

One of these methods is the mid-square method, wherein an n digit number is squared to produce a $2n$ digit number from which some middle n digits are taken for the next number and the process is repeated. The main drawback to this method is that undetected short cycles can occur. FORSYTHE [1], exhibits several of these degenerate cases and indicates the probability with which each occurs.

In any practical scheme for generating random numbers it is impossible to avoid cycling. However, cycles as such are not necessarily undesirable; for a sequence of 10 digit numbers that cycled with a period of 10^8, say, would provide an adequate supply of random digits before any number was repeated.

3. The method of congruences. The method of congruences, first proposed by LEHMER [5], provides just this sort of sequence in which no undetected cycles can occur. Lehmer suggested generating a sequence $\{x_n\}$ from the recursion relation

$$x_n \equiv kx_{n-1} \bmod m. \tag{2}$$

When $k = 23$ and $m = 10^8 + 1$, one can obtain 5,882,352 eight digit numbers before the sequence begins to repeat. It follows at once from elementary number theory that as long as k is chosen relatively prime to m, the first number in the sequence to be repeated must be the starting number x_0. Thus, cycling is easy to detect in a machine calculation of such a sequence.

Since the IBM 701 operates in the binary number system and carries numbers of 35 bits, we choose for our recursion relation

$$\begin{aligned} x_n &\equiv 23x_{n-1} \bmod (2^{35} + 1) \\ x_0 &= 10{,}987{,}654{,}321. \end{aligned} \tag{3}$$

The reduction modulo $2^{35} + 1$ can be accomplished by dropping all bits above the 35th in the product and subtracting this dropped portion from the remaining 35 bits. This can be done in a very few operations and the calculation of random numbers proceeds at high speed. A total of 172,900 of the 35 bit numbers were calculated and stored on tape in blocks of 100 in less than 20 minutes. Each number was checked for random machine errors by performing the calculation twice and was compared with x_0 to check for cycling. A subsequent calculation verified that the cycle had a period of 1,034,040, the maximum possible for the modulus $2^{35} + 1$.

JUNCOSA [2] used a congruence method to generate random numbers but chose moduli of the form 2^β. Lehmer [6] points out that the low order digits are subject to short periods and suggested moduli of the form $2^\beta \pm 1$. However, Juncosa's choice of a large value of k is superior to the choice $k = 23$ as he thus avoids several successive small numbers when some x is very small or several successive large numbers when x is very large.

4. Analysis of random numbers. To test the randomness of a table of decimal digits KENDALL and SMITH [4] used four tests which they called the frequency test, serial test, poker test, and gap test. Because our numbers were generated as binary numbers and were to be used as binary numbers, we preferred to make the tests on the binary digits. The three tests actually performed on the binary digits are analogous to the first three tests proposed by Kendall and Smith. The gap test was not made because it could not be programmed to run on the machine at the same time that the other three were run.

The problem for which the random numbers were generated required 10 bit random numbers; so the analysis was carried out entirely on the 10 bit pieces of the 35 bit numbers, two successive 35 bit numbers making seven 10 bit numbers. Only the first 112,000 35 bit numbers were tested. The numbers were analyzed in blocks of 14,000 ten bit numbers using the following three tests.

Test I was merely a count of the number of times each of the 1024 different 10 bit numbers appeared in a block of 14,000. In block 19, which we have taken as a typical block, the minimum occurrence of any number was 5, and the maximum was 26.

Test II was simply a count of ones; and in block 19 there were 69,829 ones out of 140,000 digits.

Test III was the so-called poker test. A count was made of each of the eleven different combinations of zeros and ones in a 10 bit number. The theoretically expected distribution was the binomial distribution.

In all the above tests the fit of the actual distribution to the theoretically expected was determined by calculating χ^2. The results of the tests for block 19 are shown in Table 1.

Table 2 summarizes the results of the three tests on each of the 28 blocks of 14,000 ten bit numbers. Certain blocks failed to pass one or more of the tests, notably block 13 which failed on both tests II and III. It should be noted here, however, that these tests are not independent; for a set of numbers which contain more ones than zeros would tend to be biased in favor of the combinations containing more ones on the poker test.

Because one or two of the blocks of 14,000 numbers fail on certain tests does not mean that the entire set of 28 blocks must be rejected. In fact, out of 28 values of P we should expect some extreme values, and might even view the entire set with some suspicion if none appeared. To test whether or not the number of extreme values of P is more than could be expected in a set of 28 such values, a χ^2 test was made on each of the three distributions of P. The results are shown in Table 2 and appear satisfactory for all three tests.

In Table 3 is tabulated the total count of ones and the distribution of poker hands for the entire set of 392,000 ten bit numbers. The latter test yields $P = .956$ showing that the fit of the actual to the expected distribution of poker hands is almost too good to be truly random.

TABLE 1

Tests for Randomness on a Typical Block of 14,000 Ten Bit Numbers

I. Analysis of counts of the 1024 different 10 bit numbers:
Expected count of each number: 13.67

$\chi^2 = 989$ $P = .7734$

II. Total number of 1's in the 140,000 bits: 69829
Expected number of 1's in the 140,000 bits: 70000

$\chi^2 = .8355$ $P = .36$

III. Distribution of Poker Hands in the 14,000 10 bit numbers:

	Observed	Expected	
10 zeros	12	13.67	
9 zeros, 1 one	149	136.72	
8 zeros, 2 ones	607	615.23	
7 zeros, 3 ones	1662	1640.63	$\chi^2 = 7.372$
6 zeros, 4 ones	2922	2871.09	
5 zeros, 5 ones	3468	3445.31	$P = .69$
4 zeros, 6 ones	2763	2871.09	
3 zeros, 7 ones	1633	1640.63	
2 zeros, 8 ones	627	615.23	
1 zero, 9 ones	143	136.72	
10 ones	14	13.67	

5. Conclusions. One conclusion we can draw from the results presented here is that the numbers as a whole are probably quite satisfactory for most purposes.

However, if a small quantity of random numbers, say 10,000, are to be used, we should avoid selecting them from a biased set such as block 13 in the above analysis.

If anything, the results of the tests are too close to the theoretically expected. Of the 28 blocks we find four values of P above the 95% level and three values below the 5% level on test II. This is more than twice as many as should occur in these regions. However, the number of blocks under consideration is small and the result of test II for the total of all 28 blocks is satisfactory. On the other hand, the distribution of the poker hands for the total of 28 blocks is sufficiently close

to the theoretical distribution to be considered borderline as far as randomness is concerned.

We should also remark here that there are more tests that could be made on these numbers. Some test analogous to the gap test proposed by Kendall and Smith should be made. Many other types of tests could also be made on these

TABLE 2

Result of the Three Tests on all 28 Blocks of 14,000 Ten Bit Numbers

Block No.	I. Word Count χ^2	P	II. Count of 1's Count	χ^2	P	III. Distribution of Poker Hands χ^2	P
1	957	.929	69999	.0000286	>.995	14.682	.15
2	989	.773	70055	.0864	.78	11.323	.45
3	1076	.121	70190	1.03	.35	18.601	.05
4	985	.799	69735	2.006	.14	13.184	.23
5	1016	.873	70018	.0093	.93	9.455	.49
6	910	.995	69987	.0048	.95	8.299	.60
7	956	.932	69750	1.786	.095	8.847	.55
8	1104	.038	69994	.00103	.975	8.431	.59
9	1031	.425	70093	.2465	.64	7.437	.69
10	1046	.302	69980	.0114	.930	13.411	.20
11	999	.699	69802	1.1173	.29	10.723	.39
12	1033	.409	69999	.0000286	>.995	9.638	.47
13	1031	.425	70666	12.641	<.005	26.345	<.005
14	1051	.264	70202	1.163	.28	7.511	.68
15	1029	.440	69648	3.531	.065	13.143	.22
16	1005	.652	69943	.0926	.76	7.856	.64
17	982	.816	70230	1.508	.23	13.646	.19
18	977	.846	69947	.0801	.77	13.055	.23
19	989	.773	69829	.8355	.36	7.372	.69
20	950	.948	70405	4.675	.029	13.419	.20
21	966	.898	70200	1.14	.29	10.046	.44
22	1090	.071	69935	.1204	.73	17.013	.075
23	1104	.038	69931	.1357	.72	2.331	.992
24	1034	.397	70017	.0082	.92	8.553	.59
25	1059	.212	69917	.1963	.66	12.721	.24
26	1094	.059	69866	.5117	.49	2.740	.985
27	931	.981	70002	.000114	.993	4.571	.92
28	1036	.382	69611	4.313	.031	14.818	.125

Distribution of χ^2 for the Three Tests

Interval for P	Expected Number	Test I	Test II	Test III
0– .2	5.6	5	4	6
.2– .4	5.6	5	8	7
.4– .6	5.6	4	1	7
.6– .8	5.6	5	7	5
.8–1.0	5.6	9	8	3
χ^2	2.713	6.641	2.000	
P	.60	.15	.72	

numbers. The reader is referred to Taussky and Todd [7] for a description of other tests that could be made as well as to the bibliography of that paper for further information on testing.

TABLE 3

Tests on Entire Set of 392,000 Ten Bit Numbers

total count of 1's	1,960,339
expected count of 1's	1,960,000

$\chi^2 = .11726$
$P = .73$

Distribution of Poker Hands

Observed	Expected
395	382.8
3,806	3,828.1
17,238	17,226.6
45,853	45,937.5
80,582	80,390.6
96,542	96,468.8
80,147	80,390.6
45,883	45,937.5
17,341	17,226.6
3,849	3,828.1
364	382.8

$\chi^2 = 3.791$
$P = .956$

Though the testing of the random numbers described here is by no means exhaustive, the numbers have been used successfully on Monte Carlo type solutions of problems on both digital and analog computing equipment.

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Evaluation of an Integral Arising in Numerical Integration Near a Logarithmic Singularity

This paper is concerned with the evaluation of

$$I = \int_0^y x^{-1}\,dx \int_0^x f(u)du \tag{1}$$

in terms of values of f. If this integral is known, the evaluation of

$$I_1 = \int_0^y f(x) \log x\,dx \tag{2}$$

is readily accomplished. For, let

$$I_2 = \log y \int_0^y f(x)dx. \tag{3}$$

If f is bounded, then

$$I = -\int_0^y f(x) \log \frac{x}{y}\,dx = -I_1 + I_2; \tag{4}$$

the two forms for I converge or diverge together. Equation (4) follows from (2) by partial integration. Now the integral in (3) can be approximated by any one of several standard methods. The approximate evaluation of (2) can therefore be achieved, once the evaluation of (1) is completed. In this note coefficients of the Lagrangian type are presented to facilitate the computation of I. A rigorous treatment of the error is also presented.

Let

$$\int_0^{rh} x^{-1}dx \int_0^x f(u)du = (rh/D_n) \sum_{j=0}^{n} \gamma_{jr}^{(n)} f_j + R_{nr}, \tag{5}$$

where the coefficients $\gamma_{jr}^{(n)}/D_n$ are such that the remainder R_{nr} vanishes whenever $f(u)$ is a polynomial in u of degree n,

$$D_n = \sum_{j=0}^{n} \gamma_{jr}^{(n)},$$

and D_n is independent of r, and f_j are the tabular values of $f(u)$ at the points $u = jh$, $j = 0(1)n$. Thus, we assume that $f(u)$ may be adequately represented by the $(n + 1)$ point Lagrangian polynomial. The interpolating polynomials are easily derived and have recently been published for $n = 1(1)10$ by SALZER [1]. The coefficients $\gamma_{jr}^{(n)}$ and D_n are then readily evaluated in a straightforward manner. These are given exactly in Table 1 for $n = 1(1)10$, $r = 1(1)(n + 1)$. It is understood that $\gamma_{jr}^{(n)} = 0$ if $j > n$, and the n superscript is omitted where no confusion will result. We note some properties of the integrating coefficients useful as a check on the computation.

$$\begin{aligned} &\gamma_{jr}^{(n)} = D_n, \quad r = j = 0 \\ &\gamma_{jr}^{(n)} = 0, \quad r = 0, \quad j = 1, 2, \cdots. \end{aligned} \tag{6}$$

TABLE 1

Values of the Integration Coefficients $\gamma_r^{(n)}$ and D_n

	$n = 1$, $D_1 = 4$		$n = 2$, $D_2 = 72$			$n = 3$, $D_3 = 288$			
	$r = 1$	$r = 2$	$r = 1$	$r = 2$	$r = 3$	$r = 1$	$r = 2$	$r = 3$	$r = 4$
$\gamma_0^{(n)}$	3	2	49	34	27	185	128	99	80
$\gamma_1^{(n)}$	1	2	28	40	36	145	184	171	160
$-\gamma_2^{(n)}$			5	2	−9	53	32	−9	−16
$\gamma_3^{(n)}$						11	8	9	32

	$n = 4$, $D_4 = 43200$				
	$r = 1$	$r = 2$	$r = 3$	$r = 4$	$r = 5$
γ_{0r}	26647	18352	14157	11632	10075
γ_{1r}	26162	30992	28422	25472	22250
$-\gamma_{2r}$	14568	9888	2808	−192	−3000
γ_{3r}	6062	4592	4122	6272	5750
$-\gamma_{4r}$	1103	848	693	368	−2125

	$n = 5$, $D_5 = 86400$					
	$r = 1$	$r = 2$	$r = 3$	$r = 4$	$r = 5$	$r = 6$
γ_{0r}	51667	35432	27297	22432	19175	16632
γ_{1r}	60459	68344	61929	55104	49375	45864
$-\gamma_{2r}$	45406	32496	15786	7936	3750	4176
γ_{3r}	28394	21904	18414	20864	21250	24624
$-\gamma_{4r}$	10341	8056	6471	4896	625	936
γ_{5r}	1627	1272	1017	832	975	4392

$n = 6$, $D_6 = 254\ 01600$

	$r = 1$	$r = 2$	$r = 3$	$r = 4$	$r = 5$	$r = 6$	$r = 7$
γ_{0r}	148 15625	101 22440	77 90355	63 98720	54 64325	47 86200	43 08815
γ_{1r}	200 21784	218 60544	196 16904	173 78304	155 55000	141 05664	126 52584
$-\gamma_{2r}$	189 66459	139 72344	81 65529	52 77504	36 99375	27 81864	13 59309
γ_{3r}	158 37296	123 31136	101 12976	100 59776	97 10000	93 11616	75 02096
$-\gamma_{4r}$	86 57349	67 86984	54 26919	43 83744	27 80625	18 29304	−1 73901
γ_{5r}	27 25176	21 41376	17 08776	14 22336	13 25400	19 12896	14 24136
$-\gamma_{6r}$	3 74473	2 94568	2 34963	1 96288	1 73125	1 03608	−6 99377

TABLE 1—*Continued*

$n = 7$, $D_7 = 1016\ 06400$

	$r = 1$	$r = 2$	$r = 3$	$r = 4$	$r = 5$	$r = 6$	$= 7$	$= 8$
γ_{0r}	580 41849	395 27584	303 93459	249 50784	212 97925	186 57504	166 57599	149 69344
γ_{1r}	886 31693	941 77408	838 43343	740 21888	661 35625	598 33728	546 53963	512 63488
$-\gamma_{2r}$	1014 99507	760 95072	487 89297	346 36032	265 44375	213 60672	175 68117	179 43552
γ_{3r}	1060 71969	830 00704	673 30539	627 82464	584 18125	543 01824	502 26519	530 69824
$-\gamma_{4r}$	773 52181	608 24096	485 86311	400 78336	307 00625	243 72576	195 22531	240 75776
γ_{5r}	365 34375	287 71200	229 62285	192 15360	170 48475	178 84800	178 27425	240 53760
$-\gamma_{6r}$	100 42449	79 13504	63 15579	52 93824	46 08125	38 25504	12 46119	27 76064
γ_{7r}	12 20651	9 62176	7 67961	6 44096	5 59375	4 87296	5 77661	30 45376

$n = 8$, $D_8 = 91445\ 76000$

	$r = 1$	$r = 2$	$r = 3$	$r = 4$	$r = 5$
γ_{0r}	51316 20253	34848 31648	26773 66143	21968 32768	18745 77625
γ_{1r}	87140 21626	90571 74016	80102 62206	70518 72256	62900 91250
$-\gamma_{2r}$	1 17150 48026	88827 82016	60163 01406	44819 01056	35715 91250
γ_{3r}	1 47066 62002	1 15385 14432	93102 77862	83797 38112	76228 26250
$-\gamma_{4r}$	1 34119 27280	1 05597 32480	84359 29680	70186 95680	57195 50000
γ_{5r}	84482 78542	66578 59072	53171 35002	44586 98752	38995 57750
$-\gamma_{6r}$	34839 12806	27464 40896	21936 66786	18411 02336	15973 28750
γ_{7r}	8470 27846	6678 03136	5334 77826	4478 70976	3882 28750
$-\gamma_{8r}$	921 46157	726 50912	580 45167	487 37792	422 35625

	$r = 6$	$r = 7$	$r = 8$	$r = 9$
γ_{0r}	16418 73888	14651 75803	13254 90688	12198 18933
γ_{1r}	56834 47296	51909 21526	47877 16096	43802 94186
$-\gamma_{2r}$	29669 01696	25333 57526	22239 27296	17086 54986
γ_{3r}	69760 46592	64248 40702	59942 99392	49764 58722
$-\gamma_{4r}$	48046 34880	41375 95280	36893 38880	23572 36080
γ_{5r}	36985 14432	35089 22242	33828 53632	22605 40062
$-\gamma_{6r}$	13887 36576	10643 77706	8588 53376	300 74166
γ_{7r}	3422 68416	3240 54346	4480 86016	2378 10006
$-\gamma_{8r}$	373 01472	340 08107	217 50272	−1656 19323

TABLE 1—*Continued*

$n = 9, \quad D_9 = 1\ 82891\ 52000$

	$r = 1$	$r = 2$	$r = 3$	$r = 4$	$r = 5$
γ_{0r}	1 01052 61771	68451 91552	52551 95193	43100 55424	36767 23375
γ_{1r}	1 88498 51867	1 92345 93728	1 69163 58249	1 48562 35520	1 32320 69375
$-\gamma_{2r}$	2 91173 30512	2 22465 46816	1 56159 38160	1 19737 66144	97507 30000
γ_{3r}	4 26835 37744	3 35326 55360	2 69816 71536	2 37827 25632	2 13299 30000
$-\gamma_{4r}$	4 67291 75170	3 68029 04704	2 94135 33078	2 45722 65472	2 05655 16250
γ_{5r}	3 68018 77694	2 89991 57888	2 31759 43722	1 94522 71616	1 69255 31750
$-\gamma_{6r}$	2 02380 39352	1 59485 08288	1 27484 49384	1 07054 54080	92789 35000
γ_{7r}	73812 90152	58165 89056	46502 91000	39057 05984	33840 05000
$-\gamma_{8r}$	16061 00929	12655 47520	10119 24171	8499 66592	7363 58125
γ_{9r}	1579 78735	1244 71744	995 37093	836 10112	724 31875
	$r = 6$	$r = 7$	$r = 8$	$r = 9$	$r = 10$
γ_{0r}	32196 47616	28725 68797	25989 49888	23777 14419	21825 76000
γ_{1r}	1 19437 96032	1 09018 88333	1 00437 15584	93178 99395	88180 40000
$-\gamma_{2r}$	82414 09152	71468 96176	63209 88160	56465 54064	56370 40000
γ_{3r}	1 93365 06624	1 77034 37360	1 63592 43776	1 51544 86992	1 55614 40000
$-\gamma_{4r}$	1 76858 89920	1 55558 24494	1 39346 45248	1 25168 26482	1 38906 40000
γ_{5r}	1 54736 49024	1 42984 78418	1 33216 74752	1 23234 34446	1 45369 76000
$-\gamma_{6r}$	81618 86592	69825 11368	60883 51744	52617 17880	72042 40000
γ_{7r}	29921 42592	27282 89816	27693 05600	27048 64104	41470 40000
$-\gamma_{8r}$	6515 04384	5880 61495	5117 83936	2260 72377	5934 40000
γ_{9r}	641 00160	577 82809	520 31488	619 23447	3684 40000

TABLE 1—*Continued*

$n = 10$, $D_{10} = 1327\ 79243\ 5200$

	$r = 1$	$r = 2$	$r = 3$	$r = 4$	$r = 5$	$r = 6$
7^{10}	723 64189 52621	489 09074 27264	375 22920 40047	307 61897 44640	262 34643 79625	229 68808 36416
7^{11}	1468 50033 86692	1475 13314 41408	1291 10727 89484	1131 47319 19360	1006 48502 72500	907 70292 47232
-7^{12}	2563 92311 42337	1969 25667 65376	1417 12562 57355	1107 39264 13824	914 16855 58125	780 95130 20352
7^{13}	4298 83795 77144	3378 89045 29920	2714 62539 43656	2361 55179 90912	2098 59440 55000	1890 83037 45024
-7^{14}	5492 56107 29670	4324 62531 07584	3457 99557 29538	2895 06683 01312	2455 62908 28750	2136 24559 69920
7^{15}	5191 84386 73344	4088 62017 77664	3269 66120 00064	2745 57934 75584	2383 88072 88000	2146 08690 57024
-7^{16}	3569 30461 25802	2810 59613 09568	2248 11049 67694	1888 33632 30720	1636 22328 41250	1444 80295 53792
7^{17}	1735 89478 25352	1366 70403 93216	1093 36716 74520	918 48017 26464	795 72025 05000	704 22954 57792
-7^{18}	566 60784 65079	446 03612 76480	356 87421 01341	299 80479 39072	259 72515 76875	229 92421 58784
7^{19}	111 47034 92860	87 73828 81024	70 20606 30228	58 98058 73152	51 09534 47500	45 23700 44160
-7^{110}	10 00010 93125	7 87016 39488	6 29796 70071	5 29104 93184	4 58367 90625	4 05833 32800

	$r = 7$	$r = 8$	$r = 9$	$r = 10$	$r = 11$
7^{10}	204 89534 91179	185 36086 50752	169 53494 85765	156 39368 24000	145 99151 65511
7^{11}	828 00854 84188	762 40271 93344	707 35067 85060	660 80305 60000	613 11415 75372
-7^{12}	683 25621 18711	608 43409 61280	548 86014 59769	502 00918 80000	425 30410 06467
7^{13}	1723 64701 76520	1586 42871 33696	1470 66994 47672	1377 12076 80000	1182 38205 52104
-7^{14}	1896 51342 25674	1709 46357 16608	1557 01643 36022	1441 34085 60000	1114 73966 17170
7^{15}	1958 66221 03104	1804 52357 89824	1672 63513 79904	1574 84092 80000	1166 34640 67904
-7^{16}	1274 09088 96198	1139 82266 32704	1030 29554 90970	955 90821 60000	566 48027 12382
7^{17}	636 45130 59576	599 79920 17920	566 82732 30984	548 43532 80000	293 27148 47832
-7^{18}	207 08481 40305	186 68586 94656	155 33317 55007	135 84382 80000	−1 25980 22811
7^{19}	40 72648 73764	37 00645 39648	35 36682 46812	47 36209 60000	14 60395 29460
-7^{110}	3 65314 55443	3 32289 67936	3 08711 82429	2 06133 52000	−17 34709 27025

As a function of r, $\gamma_r^{(n)}$ is a polynomial in r of degree n and so the $(n+1)$th difference must vanish. Hence

$$\Delta^{(n+1)}\gamma_r^{(n)}(r) = 0. \tag{7}$$

Thus, employing tables of Lagrangian interpolating coefficients [2], the present entries are easily extrapolated.

Treatment of the error follows the discussion given by MILNE [3, 4]. Let $c = \max(nh, rh)$. We have

$$R_{nr} = \int_0^c G_{nr}(s) f^{(n+1)}(s)\,ds, \tag{8}$$

$$n!G_{nr}(s) = \int_0^{rh} x^{-1}dx \int_0^x \overline{(u-s)}^n du - (rh/D_n)\sum_{j=0}^{n} \gamma_r^{(n)} \overline{(jh-s)}^n, \tag{9}$$

where

$$\left.\begin{aligned} \overline{(x-s)}^n \\ \end{aligned}\right\} \begin{aligned} &= (x-s)^n \quad \text{if} \quad x \geq s \\ &= 0 \quad \text{if} \quad x \leq s. \end{aligned} \tag{10}$$

Thus $(0, c)$ is the open interval in which $G_{nr}(s)$ does not vanish identically. $G_{nr}(s)$ is said to be definite if it does not change sign in this interval, otherwise it is indefinite. If $G_{nr}(s)$ is definite, the mean value theorem applies and

$$R_{nr} = E_{nr} f^{(n+1)}(\xi); \quad E_{nr} = \int_0^c G_{nr}(s)\,ds, \tag{11}$$

$$0 < \xi < c.$$

$$(n+1)!E_{nr} = rh^{n+2}\left\{r^{n+1}/(n+2)^2 - \sum_{j=1}^{n} \gamma_r^{(n)} j^{n+1}/D_n\right\}. \tag{12}$$

The latter is obtained by putting $f(x) = x^{n+1}/(n+1)!$ in (5). If we put $f(x)$ equal to the polynomial of degree $n+1$ which vanishes at $x = 0, h, \cdots, nh$ and which takes the value h^{n+1} at $x = (n+1)h$ and use equations (5), (11), and (5) with n replaced by $n+1$, we see that

$$E_{nr} = (rh/D_{n+1})h^{n+1}\gamma_{n+1,r}^{(n+1)}, \tag{13}$$

since $R_{n+1,r} = 0$. Thus, except for the case $n = 10$, the error coefficients are available in Table 1. The elements for $n = 10$ are given in Table 2. These were obtained by calculating $\gamma_{11,r}^{(11)}$ and checked using (12). They are also interpolable, recognizing that $E_{10,r}/r^2$ is a polynomial in r of degree 10. Since $\lim_{r\to 0} E_{10,r}/r^2 = 1/44$ if h is unity, there are 12 entries, and the requirement that the eleventh difference vanish furnished a further check on the entries. The values were calculated to 7D and rounded to 5D for presentation.

The error given by (8) may also be appraised by

$$|R_{nr}| \leq \sup|f^{(n+1)}(s)| \int_0^c |G_{nr}(s)|\,ds. \tag{14}$$

Alternatively R_{nr} can be expressed in terms of an integral on one of the lower derivatives $f^{(n)}, f^{(n-1)}, \cdots, f^{(1)}$ of f. The only essential advantage of definiteness is that it simplifies the evaluation of the integral in (14).

TABLE 2

Values of Error Coefficients for $n = 10$

r	E_{10r}/h^{12}
1	0.00666
2	0.01046
3	0.01257
4	0.01408
5	0.01524
6	0.01620
7	0.01701
8	0.01770
9	0.01834
10	0.01873
11	0.02454

Concerning the definiteness of $G_{nr}(s)$, general theorems are wanting, but in each case definiteness can be checked directly. It has been established that the function is definite for all reported values of r and n such that $r \leq n$. If $r = n + 1$, $G_{n,n+1}(s)$ is definite (indefinite) if n is even (odd). In the indefinite cases, $G_{n,n+1}(s)$ has one zero in the open interval $(0, n + 1)$. Let ζh be the vanishing point. Then the remainder can be decomposed into two parts. Thus,

$$(15) \qquad R_{n,n+1} = f^{(n+1)}(\xi_1) \int_0^{\zeta h} G_{n,n+1}(s)ds + f^{(n+1)}(\xi_2) \int_{\zeta h}^{(n+1)h} G_{n,n+1}(s)ds$$

where $0 < \xi_1 < \zeta h < \xi_2 < (n + 1)h$. The values of ζ and the two integrals in (15) are recorded in Table 3 for $n = 1(2)9$, mostly to 5d. The zeros were evaluated by inverse interpolation using TAYLOR's theorem. Availability of the derivatives facilitated the calculation of the integrals.

The heuristic verification that $G_{nr}(s)$ is definite is straightforward, but the numerical work is considerably simplified, especially for n large, using some results due to BARRETT [5]. Let $G_{nr}^{(k)}(s)$ denote the k-th derivative of $G_{nr}(s)$ with respect to s. He has shown that $G_{nr}^{(k)}(s)$ is defined and continuous in $(0, c)$ closed for $0 \leq k \leq n - 1$. With this restriction on k, $G_{nr}^{(k)}(0) = G_{nr}^{(k)}(c) = 0$. If $\nu(k)$ denotes the number of zeros of $G_{nr}^{(k)}(s)$ in $(0, c)$ closed, then $\nu(k) \geq k + 2$ and $G_{nr}(s)$ is definite if equality holds. If $k = 0$, the condition is both necessary and sufficient. For the problem at hand, we first examined the function $G_{nr}^{(n-1)}(s)$ for $n \geq 5$. If $\nu(n - 1) = n + 1$, then $G_{nr}(s)$ is definite. In a few cases, it was found that $\nu(n - 1) = n + 2$, and in some of these definiteness was established by showing that $\nu(n - 2) = n$. In the remaining cases, $\nu(n - 2) = n + 1$, and these were

TABLE 3

Values of Zeros of $G_{n,n+1}(s)$ and Error Coefficients for the Indefinite Cases

n	ζ	$\int_0^{\zeta h} G_{n,n+1}(s)ds/h^{n+2}$	$\int_{\zeta h}^{(n+1)h} G_{n,n+1}(s)ds/h^{n+2}$
1	0.84829	−0.08602	0.03046
3	2.02495	−0.03961	0.00554
5	3.22618	−0.02661	0.00214
7	4.46830	−0.02001	0.00098
9	5.71960	−0.01602	0.00050

shown to be indefinite by examining $G_{sr}(s)$ itself. From the computational point of view, examination of the higher derivatives facilitates the study as the inconvenience of handling large powers of s is removed.

KAPLAN [6] has given coefficients to facilitate the evaluation of

$$\int_{r_1 h}^{r_2 h} g(x)dx \tag{16}$$

$$g(x) = x^\eta[A(x)\log x + B(x)], \quad \eta = 0, 1$$

in terms of values of $g(x)$. He writes

$$\int_{r_1 h}^{r_2 h} g(x)dx \sim h \sum_{i=\rho}^{\rho+P} c_i g(ih) \tag{17}$$

where ρ and $\rho + P$ are number of tabular intervals h from the singularity to the first and last of group of $P + 1$ consecutive ordinates. All formulas are derived on the basis that $A(x)$ and $B(x)$ can be represented by quadratic polynomials. For the most part, the tables can be described as follows. Coefficients c_i are given for $\rho = 1, 5, 10$ or 15 and $(r_1, r_2) = (\rho + s, \rho + s + 1)$, $s = 0(1)4$; also for $\rho = 1$ and $(r_1, r_2) = (0, 1)$, $(0, 5)$ and $(0, 6)$. Here $P = 5$. If $\eta = 0$, additional entries are presented for $\rho = 1$, $P = 4$ and $(r_1, r_2) = (s, s + 1)$, $s = 0(1)4$. The latter group are five term formulas and it is assumed that the constant term in the representation for $A(x)$ is missing. Thus, tabular values at the origin are never used. All entries are to 10D. The error term is not considered.

The principal application of Kaplan's table occurs when $A(x)$ and $B(x)$ are not individually known, but $g(x)$ only is tabulated. If each is known, then tables in present paper are more advantageous since representation of $f(x)$ by higher order polynomials is possible. Present tables are useful to integrate out to ten tabular intervals h from the singularity after which ordinary integration formulas may be employed. For integration beyond $10h$, recourse to Kaplan tables may still be appropriate depending on the nature of $f(x)$.

The problem treated in this note is an example of approximate product integration, and has been studied by BEARD [7]. Other examples have been given by SHELDON [8] and LUKE [9, 10].

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On Irregular Negative Determinants of Exponent 9*n*

It may first be stated that this article is concerned only with the irregular negative determinants which occur in the study of quadratic forms. (The technical terms used below, *e.g.*, "determinant," "irregular," "exponent," "duplicating," "opposite," and "genera II, IV, VIII, etc.," are fully explained in Mathews, "Theory of Numbers" (Part I., 1892) on pages 59, 178, 182, 149, 68, and 132 respectively [1].)

Only three examples of the above determinants have as yet been given in mathematical literature, *viz.*, $-D = 3299$ (SCHOLZ and TAUSSKY) [2], 6075 (PEPIN) [3], and 11907 (GAUSS) [4]. The writer, in his Thesis for the London Ph.D. degree written in 1953, gives 55 more examples, the complete table of 58 determinants up to the limit 150,000 being:

$-D =$			
3299	56403	94284	117612
6075	58563	97200	120987
11907	60075	99387	122715
17739	64395	101675	128547
23571	70227	103627	132300
24300	70956	105219	134379
27675	75411	107163	135675
29403	76059	107419	140211
33075	77571	108675	140940
35235	81675	109907	141075
41067	81891	110700	146043
46899	85995	111051	148419
47628	87075	114075	149499
52731	87723	115425	
54675	93555	116883	

A characteristic property of the determinants of exponent 9 given in the above list (with the exception of -3299) is that they possess 13 classes each of which when duplicated produces its own opposite. These may be termed "critical classes."

For the determinant $-D = 6075$ they are:

4	1	1519
9	3	676
36	3	169
36	15	175
76	9	81
79	18	81
81	36	91
19	9	324
25	5	244
31	1	196
49	1	124
61	5	100
76	29	91

In the exceptional case, $-D = 3299$, there are only 4 critical classes, the single genus it possesses having 81 classes, which tabulate into 9 periods of 9 classes each. All the other members of the list of 58 determinants given above have genera II, IV, VIII, etc.

In an attempt to find a relation between some of these determinants it was found that 25 of them were of the form $-D = 6075 + 5832p$, *viz.*, 6075, 11097, 17739, 23571, etc. Another series was found to be of the form $27675 + 27000p$, giving 27675, 54675, 81675, etc., [5].

In these series, when the critical classes (a, b, c) are arranged so that a is a perfect square, it is found that each member gives the same values of a and b; *e.g.*, in the above example, 6075, the critical classes can be shown thus:

$-D = 6075$				$-D = 11907$	$-D = 17739$
a	b	c		c	c
4	1	1519		2977	4435
9	3	676		1324	1972
36	3	169		331	493
36	15	175		337	499
81	9	76		148	220
81	18	79		151	223
81	36	91	for the next two	163	235
324	9	19	members 11907,	37	55
324	45	25	17739 the	43	61
324	63	31	c-values are:	49	67
324	99	49		67	85
324	117	61		79	97
324	153	91		109	127

and for all the others the c-values increase respectively in arithmetical progression.

It is found that the determinants (with the exception of $-D = 3299$) can be arranged in 20 series as follows:

Series No.	Form	Members
1	$6075 + 5832p$	6075, 11907, 17739, 23571, 29403, 35235, 41067, 46899, 52731, 58563, 64395, 70227, 76059, 81891, 87723, 93555, 99387, 105219, 111051, 116883, 122715, 128547, 134379, 140211, 146043.
2	$27675 + 27000p$	27675, 54675, 81675, 108675, 135675.
3	$33075 + 27000p$	33075, 60075, 87075, 114075, 141075.
4	$24300 + 46656p$	24300, 70956, 117612.
5	$47628 + 46656p$	47628, 94284, 140940.
6	$75411 + 74088p$	75411, 149499.
7	$11907 + 74088p$	11907, 85995. (11907 also appears in Ser. 1)
8	$33075 + 74088p$	33075, 107163. (33075 also appears in Ser. 3)

Series No.	Form	Members
9	$132300 + 74088000p$	132300
10	$97200 + 1728000p$	97200
11	$115425 + 91125p$	115425
12	$110700 + 216000p$	110700
13	$101675 + 343000p$	101675
14	$77571 + 474552p$	77571
15	$56403 + 580093704p$	56403
16	$103627 + 170584511336p$	103627
17	$107419 + 7066834559000p$	107419
18	$109907 + 982107784p$	109907
19	$120987 + 64964808p$	120987
20	$148419 + 35937000p$	148419.

In every case the factor of p (*i.e.*, the constant difference) is the cube of the product of the square roots of the a-values of the 1st, 2nd, and 5th classes.

All the critical classes of these determinants have been tested by duplication, and their opposites were obtained in every case.

As a further test with a determinant beyond our assumed limit, we may take $-D = 187596$, the 4th member of Series 5, and we have:

$$-D = 187596$$

a	b	c
9	3	20845
16	2	11725
144	30	1309
144	66	1333
81	9	2317
81	18	2320
81	36	2332
1296	414	277
1296	558	385
1296	18	145
1296	306	217
1296	126	157
1296	450	301

the last six of which reduce to:

277	137	745
385	173	565
145	18	1296
217	89	901
157	31	1201
301	149	697.

Choosing one of these classes, say (301, 149, 697), we can duplicate it by compounding (301, 149, 697) with (697, −149, 301).

This is done most simply by a method devised by the writer:

Let the classes be $(a, b, \cdots)$ and $(a', b', \cdots)$ where a and a' are prime to each other, and take p and q such that either ap and $a'q$, or $a'p$ and aq differ by unity. This can nearly always be done mentally, but when a and a' are not small, the values of p and q are more quickly found as the constituents of the penultimate convergent of the continued fraction representing a/a' or a'/a. The required compound class is then given by $(aa', a'bp + ab'q, \cdots)$ or by $(aa', a'bq + ab'p, \cdots)$, care being taken that the signs of p and q are so chosen that the smaller of the two products, *e.g.*, aq and $a'p$, say, shall be negative. Applying this to the case in hand, we get:

$$
\begin{aligned}
&(697.301, 697.149(-19) + 301(-149)44, \cdots),\\
\text{or}\quad &(209797, -1973207\ -1973356, \cdots),\\
&(209797, -3946563, \cdots),\\
&(209797, +39580, 7468),\\
&(7468, 2240, 697),\\
&(697, 149, 301),\\
&(301, -149, 697),
\end{aligned}
$$

which shows that (301, 149, 697) is a critical class, and each of the twelve other classes when similarly tested is found to be a critical class.

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3. T. PEPIN, *Atti. Acad. Pont. Nuovi Lincei*, 33, 1881, p. 354–391.
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5. The 11th member of the second series, *i.e.* −297675, has exponent 27, with 40 critical classes.

TECHNICAL NOTES AND SHORT PAPERS

Selected References on Use of High-Speed Computers for Scientific Computation

The author is often asked to recommend reading to orient mathematicians in the impact of high-speed computers on numerical analysis. The following list was prepared in answer to one such request, but does not pretend to be definitive. The author is indebted to C. B. TOMPKINS for several suggestions.

For a list of books not necessarily influenced by high-speed computers, but highly pertinent to their use, see G. E. FORSYTHE, "A numerical analyst's fifteen-foot shelf," *MTAC*, v. 7, 1953, p. 221–228.

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Mathematical Reviews (Numerical and Graphical Methods section)
Mathematical Tables and Other Aids to Computation
Naval Research Logistics Quarterly
Proceedings of the Association for Computing Machinery (terminated)
Proceedings of the Cambridge Philosophical Society
Quarterly of Applied Mathematics
Quarterly Journal of Mechanics and Applied Mathematics
Vychislitel'naîa Matematika i Vychislitel'naîa Tekhnika (Moscow)
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Modified Quotients of Cylinder Functions

The name in the title of this note is applied to the function, $\mathfrak{C}_\nu(z)$, defined by the following equation,

$$\mathfrak{C}_\nu(z) = \frac{zC_{\nu-1}(z)}{C_\nu(z)} \tag{1}$$

where $C_\nu(z)$ is a cylinder function [1] which satisfies the pair of recurrence formulae,

$$\frac{2\nu}{z} C_\nu(z) = C_{\nu-1}(z) + C_{\nu+1}(z) \tag{2}$$

$$2C_\nu'(z) = C_{\nu-1}(z) - C_{\nu+1}(z)$$

$\mathfrak{C}_\nu(z)$ has not repeated zeros and poles with possible exception of the origin and satisfies the following RICCATI's equation,

$$\frac{dy}{dz} + \frac{1}{z}(y^2 - 2\nu y) + z = 0. \tag{3}$$

Its derivative,

$$\frac{\partial \mathfrak{C}_\nu(z)}{\partial z} = \frac{z}{C_\nu^2(z)} \{C_{\nu-1}(z)C_{\nu+1}(z) - C_\nu^2(z)\} \tag{4}$$

has a close relation to the famous integral due to LOMMEL,

$$\int zC_\nu^2(z)dz = \left[\frac{z^2}{2}\{C_\nu^2(z) - C_{\nu-1}(z)C_{\nu+1}(z)\}\right]. \tag{5}$$

Introducing each of the three kinds of Bessel functions, $J_\nu(z)$, $Y_\nu(z)$, $H_\nu^{(1)}(z)$ and $H_\nu^{(2)}(z)$, into the equation (1) in place of $C_\nu(z)$, we obtain corresponding kinds of modified quotients, $\mathfrak{J}_\nu(z)$, $\mathfrak{Y}_\nu(z)$, $\mathfrak{H}_\nu^{(1)}(z)$ and $\mathfrak{H}_\nu^{(2)}(z)$, respectively. In various boundary value problems of mathematical physics, we encounter quite often the Bessel functions in quotient forms [2]. It is obvious that the modified quotients defined here give a convenient approach to mathematical analysis and numerical estimates of these problems. Moreover there is a remarkable parallelism between the modified quotients and the trigonometric cotangent and tangent, as there is between the Bessel functions and the sine and cosine. Therefore the modified quotients should have the same *raison d'être* in cylinder functions as the cotangent and tangent in trigonometric functions.

From the above consideration it seems highly desirable to give permanent symbols to these modified quotients, to collect their formulae and to construct their tables. To this end an attempt was made [3], but further cooperation and criticism of interested workers are necessary.

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2. For example:
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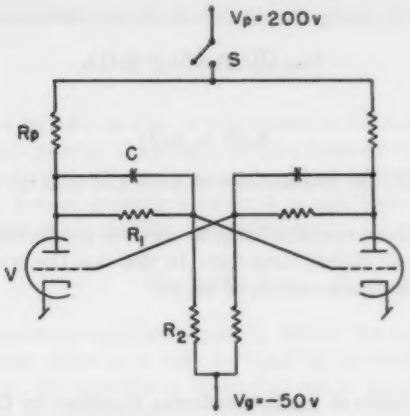
Flip-flop as Generator of Random Binary Digits

The aim of the present note is to show that a well known electronic element of digital computers, the flip-flop, may be used for generating a series of random binary digits with equal probabilities.

Let us consider a flip-flop as shown on fig. 1 and let A and B denote two possible stable states of the flip-flop. If we switch on the contact S, the flip-flop will be randomly set in one of its states A or B. We may obtain by the aid of the flip-flop a sequence of $2k$ random elements $X_1, X_2, \cdots, X_{2k}$, (abbreviated $\{X_{2k}\}$), where

$$X_j = \begin{cases} \text{A, if } j\text{-th switching on the contact } S, \text{ set flip-flop in state A} \\ \text{B, if } j\text{-th switching on the contact } S, \text{ set flip-flop in state B} \end{cases}$$

and $1 \leq j \leq 2k$.



$R_p = 10\ K\Omega$
$R_1 = 160\ K\Omega$
$R_2 = 50\ K\Omega$
$C = 50\ \mu\mu fd$
$V = \frac{1}{2}$ 6SN7

FIG. 1.

In this way we may obtain a finite random series of A and B which are statistically independent. One series produced by the aid of a flip-flop is given below:

AABAABBABBABBBABBAABABABBABAABABABBAA
BABABBBBBBBBBABBBABAABBB.

Let $\{Y_k\}$ be the sequence of k pairs of elements of $\{X_{2k}\}$ such that $Y_i = X_{2i-1}, X_{2i}$, where $1 \leq i \leq k$. Omitting in $\{Y_k\}$ all elements of the form AA and BB we obtain a third sequence whose elements are the pairs AB and BA only, denoted in the following by 0 and 1 respectively.

Let $p_j(\mathrm{A})$ and $p_j(\mathrm{B})$ denote probabilities that j-th switching on of contact S set flip-flop in state A or B respectively and suppose that $p_j(\mathrm{A})$ and $p_j(\mathrm{B})$ are asymmetric, say $p_j(\mathrm{A}) > p_j(\mathrm{B})$. Supposing that the flip-flop does not change its properties during two successive switchings, we may write

$$p_{2i-1}(\mathrm{A}) = p_{2i}(\mathrm{A}) \tag{1}$$

$$p_{2i-1}(\mathrm{B}) = p_{2i}(\mathrm{B}). \tag{2}$$

From 1 and 2 we have

$$p_{2i-1}(\mathrm{A}) \cdot p_{2i}(\mathrm{B}) = p_{2i-1}(\mathrm{B}) \cdot p_{2i}(\mathrm{A}). \tag{3}$$

Because

$$p_{2i-1}(\mathrm{A}) \cdot p_{2i}(\mathrm{B}) = p_i(0) \tag{4}$$

and

$$p_{2i-1}(\mathrm{B}) \cdot p_{2i}(\mathrm{A}) = p_i(1), \tag{5}$$

therefore

$$p_i(0) = p_i(1) \tag{6}$$

where $p_i(0)$ and $p_i(1)$ are probabilities of zeros and ones in the i-th place of the sequence $\{Y_k\}$.

The procedure above described may be used for production of binary random numbers by automatic digital computers. In this case the manual switch S must be replaced by an electronic switch, of course.

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Numerical Solution of the Schroedinger Equation for Central Fields

A fast program has been written for the ILLIAC to integrate the radial SCHROEDINGER equation

$$u''(r) - [l(l+1)/r^2 + 2V(r) - E]u = 0 \tag{1}$$

$$\text{boundary conditions: } u(0) = 0 \quad u(r) \text{ bounded}$$

for any well-behaved potential $V(r)$. More generally the program can integrate any linear second-order differential equation which can be put in the form

$$u''(r) - g(r)u(r) = q(r) \tag{2}$$

with $q(r)$ vanishing at zero and infinity, $r^2g(r)$ bounded at zero, and $g(r)$ bounded for large r [1].

A distinctive feature of the program is the use of the NOUMEROV [2], [3] method for the integration. This is faster than techniques (such as the RUNGE-KUTTA) which depend on making an estimate of one or more forward points and improving this by an iteration scheme. Here there is no iteration, but the error in the "estimate" of each forward point is of eighth order in the step-size h, so that the truncation error may still be kept small. The essence of the method is the elimination of all odd powers of h from the TAYLOR expansion about any point by working with three points instead of two, followed by a change of dependent variable which removes the h^4-term. The calculation of a forward point to order h^8 thus requires the value of the dependent variable at the six preceding points.

For an equation in the form (2), the required new dependent variable is

$$y = u - (h^2/12)(gu + q). \tag{3}$$

The prescription for calculating forward points is

$$y_{n+1} = 2y_n - y_{n-1} + h^2 \frac{g_n y_n + q_n}{1 - \frac{1}{12}h^2 g_n} - \frac{1}{240}\delta^6 y_n, \tag{4}$$

where the sixth central difference in the last term is given, to order h^8, by

$$\delta^6 y_n = 3\delta^6 y_{n-2} - 2\delta^6 y_{n-3}. \tag{5}$$

This prescription can be used as soon as y is known at the first six points.

To get this far, *i.e.*, to start the integration, an iteration method is used: The difference equation (4) without the last term is solved by successive approximations at the first six points, guessing a value of y_0 and improving this guess by applying the condition (seventh order approximation)

$$\delta^6 y_2 = \delta^6 y_3. \tag{6}$$

For the inhomogeneous equation ($q \not\equiv 0$), where the normalization is not arbitrary, the starting value $u_1 = u(h)$ is found by another iteration scheme: With an estimated u_1, the equation is integrated out to some large r, where the inhomogeneity q has become negligible. Application of the condition that u be small and monotonically decreasing leads to an improved estimate of u_1. This scheme, which seldom requires more than three iterations, is practical because of the speed of integration. (ILLIAC takes less than 40 milliseconds per h-step.)

In its present form, the program can—

—find eigenvalues E (permitted negative energies)
—print out values of the wave functions $u = u_E(r)$ at desired values of r, and find the nodes and extrema
—evaluate matrix elements $\int u_{E_i} u_{E_j}$ (with ILLIAC's electrostatic memory of 2^{10} words, at 40 bits per word, there is enough space to integrate several wave-functions simultaneously)
—solve self-consistent field problems (Schroedinger equation and POISSON equation solved simultaneously).

A "guide" to the program has been prepared, containing a detailed description of the mathematical method and instructions for use of the existing tapes, and is available in mimeographed form at the University of Illinois Digital Computer Laboratory.

The authors are indebted to Professors JOHN BLATT and J. N. SNYDER for helpful suggestions.

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1. If $q(r)$ is not identically zero, i.e., for the inhomogeneous equation, there is the additional restriction that $r^2 g(r)$ be bounded everywhere.
2. G. W. PRATT, JR., "Wave functions and energy levels for Cu^+ as found by the Slater approximations to the Hartree-Fock equations," *Phys. Rev.*, 88, 1952, p. 1217.
3. B. NOUMEROV, Monthly Notices *Roy. Astron. Soc.*, 84, 1924, p. 592.

Note on Using the Reciprocal Function for a Linear Inverse Interpolation

Let it be required to find the argument, x, by linear interpolation from a table of values of $y(x)$. If there is also a table of values of the reciprocal function $\frac{1}{y(x)}$, then a better linear interpolation may result by using the reciprocal function. Frequently, in applications such as in triangle or compound interest problems, $y(x)$ is given as equal to a quotient $\frac{A}{B}$. Hence the solution of $\frac{1}{y(x)} = \frac{B}{A}$ for x by linear interpolation can be made with equal facility. For example, if $\tan x = \frac{A}{B}$ is given for determination of x, we might just as easily determine x from the relation $\cot x = \frac{B}{A}$, using the $\cot x$ table. It is the purpose of this note to determine the conditions for which interpolation will be improved by using the reciprocal function.

Suppose that $y(x)$ and $z(x)$ are two monotonic functions such that $yz = 1$. Since the slopes of y and z have the relation, $z'y + y'z = 0$, and hence are always opposite in sign, we choose y as the increasing function.

Now the value, x_a, given by linear interpolation for the argument, x, in a table of values of $y(x)$ is given by

$$x_a = x_1 + \frac{y - y_1}{y_2 - y_1}(x_2 - x_1) \quad \text{where} \quad \begin{cases} x_1 < x_a < x_2 \\ y_1 < y < y_2. \end{cases}$$

(We assume that the function $y(x)$ is positive. Roundoff errors are not considered.) The error, E_a, due to the interpolation is given by $E_a = x_a - x$. See [1] for the maximum error inherent in determining a function by linear interpolation from tables. It is given by

$$|\text{error}| \leq \frac{(x_2 - x_1)}{8} M,$$

where M denotes the maximum absolute value of $f''(x)$ in the interval (x_1, x_2). However, the problem presented in this paper is one of inverse interpolation. Hence this error formula can be applied only after the function is inverted. For compound interest functions, see [2].

The corresponding interpolated value, x_b, and the error, E_b, may be determined for the reciprocal function, z. (See [3]. It should be noted that this paper is not considering reciprocal differences. We consider only the differences of the reciprocal function.) The condition that the interpolation is improved by using the reciprocal function is given by $|E_b| < |E_a|$. We note that $E_a < 0$ (>0) if $y'' > 0$ (<0) and $E_b > 0$ (<0) if $z'' > 0$ (<0) and thus we have the following cases:

Case I. $y'' < 0$ and $z'' > 0$, then E_a and E_b are both positive. In this case the result is poorer since always

$$(1)\quad E_b \equiv x_1 - x + \frac{z - z_1}{z_2 - z_1}(x_2 - x_1) > x_1 - x + \frac{y - y_1}{y_2 - y_1}(x_2 - x_1) \equiv E_a.$$

The difference in the errors is given by

$$(2)\quad E_b - E_a = \frac{(y - y_1)(y_2 - y)(x_2 - x_1)}{(y_2 - y_1)y}.$$

Case II. $y'' > 0$ and $z'' < 0$, then E_a and E_b are both negative. Inequality (1) remains true, but now $|E_b| < |E_a|$ and hence the interpolation is always improved.

Case III. $y'' < 0$ and $z'' < 0$. This case is impossible. The reason for this is as follows: From $yz = 1$, one obtains $z'y + y'z = 0$ and $z''y + 2z'y' + zy'' = 0$. Hence $z'' + z^2y'' = 2(z')^2y$, and since $y > 0$, y'' and z'' cannot both be negative. In fact, if y'' (or z'') is negative, then z'' (or y'') is positive.

Case IV. $y'' > 0$ and $z'' > 0$, then E_a is negative and E_b is positive. The inequality is $E_b < |E_a|$, giving the condition,

$$(3)\quad E_b - |E_a| = \frac{y - y_1}{y_2 - y_1}(x_2 - x_1)\left(1 + \frac{y_2}{y}\right) - 2(x - x_1) < 0$$

or

$$(4)\quad E_b - |E_a| = \frac{z_1 - z}{z_1 - z_2}(x_2 - x_1)\left(1 + \frac{z_2}{z}\right) - 2(x - x_1) < 0.$$

Some examples follow:

I $\begin{cases} y = \sec x \left(0 \leqq x < \frac{\pi}{2}\right) & y'' > 0 \quad \text{Case II} \\ z = \cos x & z'' < 0 \quad \text{Result is always improved.} \end{cases}$

II $\begin{cases} y = \sin x \left(0 < x \leqq \frac{\pi}{2}\right) & y'' < 0 \quad \text{Case I} \\ z = \csc x & z'' > 0 \quad \text{Result is always poorer.} \end{cases}$

III $\begin{cases} y = \sqrt{x}\ (x > 0) & y'' < 0 \quad \text{Case I} \\ z = \left(\frac{1}{\sqrt{x}}\right) & z'' > 0 \quad \text{Result is always poorer.} \end{cases}$

IV $\begin{cases} y = e^x\ (x > 0) & y'' > 0 \quad \text{Case IV} \\ & \quad E_b - |E_a| = 0 \text{ if } x = \frac{x_1 + x_2}{2} \\ z = e^{-x} & z'' > 0 \quad \text{Result is improved if} \\ & \quad \frac{x_1 + x_2}{2} < x < x_2. \end{cases}$

V
$$\begin{cases} y = (1+i)^n \text{ for } n & y'' > 0 \quad \text{Case IV} \\ & E_b - |E_a| = 0 \text{ if } n = \dfrac{n_1+n_2}{2} \\ z = (1+i)^{-n} \ (n>0) & z'' > 0 \quad \text{Result is improved if} \\ & \dfrac{n_1+n_2}{2} < n < n_2. \end{cases}$$

VI
$$\begin{cases} y = s_{\overline{n}|}(i) \text{ for } n & \text{Case IV} \\ x = \dfrac{1}{s_{\overline{n}|}}(i) \ n > 0 & \text{Result is same as for } (1+i)^n. \end{cases}$$

VII
$$\begin{cases} y = a_{\overline{n}|}(i) \text{ for } n & y'' < 0 \quad \text{Case I} \\ z = \dfrac{1}{a_{\overline{n}|}}(i) \ (n>0) & z'' > 0 \quad \text{Result is always poorer.} \end{cases}$$

VIII
$$\begin{cases} y = \tan\theta \text{ for } \left(0 < \theta < \dfrac{\pi}{2}\right) & y'' > 0 \quad \text{Case IV} \\ z = \cot\theta & z'' > 0. \end{cases}$$

The error in this example is given by

$$|E_a| = \left|\theta_1 + \frac{\tan\theta - \tan\theta_1}{\tan\theta_2 - \tan\theta_1}(\theta_2 - \theta_1) - \theta\right|.$$

In this case $|E_a|$ increases as θ increases from 0 to $\dfrac{\pi}{2}$ for constant values of $\theta_2 - \theta_1$ and $\theta - \theta_1$. That this is true is seen by the relation

$$\frac{\tan\theta - \tan\theta_1}{\tan\theta_2 - \tan\theta_1} = \frac{\sin(\theta - \theta_1)\cos\theta_2}{\sin(\theta_2 - \theta_1)\cos\theta}.$$

Now

$$|E_a(\theta, \theta_1, \theta_2)| = E_b\left(\frac{\pi}{2} - \theta, \frac{\pi}{2} - \theta_1, \frac{\pi}{2} - \theta_2\right).$$

Hence $E_b < |E_a|$ only if $\theta > \dfrac{\pi}{4}$, and in this case the interpolation is improved; but if $\theta < \dfrac{\pi}{4}$ the interpolation is poorer.

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1. J. B. SCARBOROUGH, *Numerical Mathematical Analysis*, Johns Hopkins Press, 1930, p. 107.
2. HUGH E. STELSON, "The accuracy of linear interpolation in tables of the mathematics of finance," *MTAC*, v. 3, 1949, p. 408–412.
3. L. M. MILNE-THOMSON, *The Calculus of Finite Differences*, Macmillan and Co., 1933, p. 104–123.
4. W. E. MILNE, *Numerical Calculus*, Princeton Univ. Press, 1949, p. 219–236.

REVIEWS AND DESCRIPTIONS OF TABLES AND BOOKS

1[A, L].—A. D. Booth, "A note on approximating polynomials for trigonometric functions," *MTAC*, v. 9, 1955, p. 21–23.

$J_n(\frac{1}{2}\pi)$ is given to 11D for $n = 0(.5)12.5$.

J. T.

2[B, C].—John von Neumann & Bryant Tuckerman, "Continued fraction expansion of $2^{\frac{1}{2}}$," *MTAC*, v. 9, 1955, p. 23–24.

Let $S_n(x)$ be the sum of the first n partial quotients of the continued fraction for x. A table of $n \log n/\log 2$ and $S_n(2^{\frac{1}{2}})$ is given for $n = 100(100)2000$; the first expression is given to 1D; the second is given exactly.

J. T.

3[D].—Birger Jansson, *Numerisk fourieranalys för hand-och hälkortsberäkning.* 44 leaves of blue line prints from transparencies, 29.7 cm., deposited in the UMT File.

This contains tables of $\sin\left(vn\frac{360}{100}\right)$ and $\cos\left(vn\frac{360}{100}\right)$ for $v = 0(1)25$ and $n = 1(1)25$ and of $\sin\left(vn\frac{360}{400}\right)$ and $\cos\left(vn\frac{360}{400}\right)$ for $v = 0(1)100$, $n = 1(1)50$, all to 4D. The tables are prepared to assist in Fourier analysis using desk calculators or punched card machinery. A remarkably detailed description of procedures for Fourier analysis included; this is based mainly on an assumption that punched card machinery similar to International Business Machines is available. The discussion leads up to a careful time estimate for the calculation.

The author announces but does not include tables of $\sin\left(vn\frac{360}{1200}\right)$ and $\cos\left(vn\frac{360}{1200}\right)$ for $v = 0(1)300$, $n = 1(1)160$. Tables designed for the same application have been prepared also by L. W. Pollak and by Owen Mock; these are reviewed in *MTAC*, v. 5, 1951, p. 19–21 and p. 149, and in *MTAC*, v. 9, 1955, p. 72 and p. 196. No earlier tables are known to contain as detailed instructions for machine analysis.

C. B. T.

4[F].—Andrew S. Anema, *A table of primitive Pythagorean triangles with their generators, having identical perimeter totals.* 13 typewritten pages deposited in the UMT File.

The main table is a set of 107 primitive Pythagorean triangles with identical perimeters. The table contains two primitive Pythagorean triangles with perimeter 1716 (sides 364, 627, and 725; 748, 195, 772), three primitive Pythagorean triangles with perimeter 14280, four primitive Pythagorean triangles with perimeter 3 17460, and 107 primitive Pythagorean triangles with perimeter 60850 05270 54420.

These were computed without aid of machines. The author, an old hand at lightning calculation, includes several comments about the method of calculation.

One example from each of the sets with 3, 4, and 107 entries follows:

From set of three:

$$P = 14280 = 2 \cdot 2^2 \cdot 3 \cdot 5 \cdot 7 \cdot 17$$
$$3 \cdot 5 \cdot 7 = 105,\ 2^2 \cdot 17 = 68 \text{ (generator } a\text{), 37 is generator } b;$$
$$A = 2ab = 5032,\ B = a^2 - b^2 = 3255,\ C = a^2 + b^2 = 5993.$$

From set of four:

$$P = 3\,17460.$$
$$A = 1\,53868,\ B = 9435,\ C = 1\,54157.$$

From set of one hundred seven:

$$P = 60850\ 05270\ 54420,$$
$$A = \ 2763\ 24098\ 54668,$$
$$B = 28977\ 68095\ 17195,$$
$$C = 29109\ 13076\ 82557.$$

C. B. T.

5[F].—N. G. W. H. BEEGER, *Table of the Least Factor of the Numbers that are not Divisible by* 2, 3, 5 *of the Eleventh Million.* 429 tables, 34 pages of form, 33½ cm., deposited in the UMT FILE.

In 1949 D. JARDEN and A. KATZ published, *Page 477 to D. N. Lehmer's "Factor Table for the First Ten Millions"* [1]. It contains the table of the least factor of all the numbers not divisible by 2, 3, 5, and 7, in the interval 10 017 000–10 038 000. I have checked this extension against the part of KULIK's table, "Magnus Canon Divisorum" [2], and against the part of a table constructed by L. POLETTI by means of his printed fasciles, *Neocribrum. No discrepancy was found.* Therefore my construction of the table of the least factors of the eleventh million was commenced at cycle 335, that is to say at 10 030 021.

In the printed fasciles (conceived by L. POLETTI), mentioned above, the least factors 7, 11, 13 are *printed.* I used "*stencils*" to insert the least factors 17, 19, 23, ···, 359. All insertions in the fasciles were made by rubber stamps. Therefore I calculated by means of a calculating machine and using J. GLAISHER's table of the differences of consecutive primes [3]: the products $367q$, $373q$, ···, $3313q$ (q are suitable primes) in the eleventh million. These were then inserted in the fasciles. The resulting table was checked: first against KULIK's table, second against the mentioned table by L. POLETTI. The discrepancies were corrected. This work confirmed the primality of all the numbers of *Liste des nombres premiers du onzième million* (*plus précisément de* 10 006 741 *à* 10 999 997), d'après les tables manuscrites de J. PH. KULIK, L. POLETTI, et R. J. PORTER. Imprimerie "Werto," Amsterdam, 1951.

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1. D. Jarden & A. Katz, *Additional page (477) to D. N. Lehmer's Factor Table, Riveon Lemat.*, v. 3, p. 50.
2. J. Ph. Kulik, *Magnus Canon Divisorum.* [See UMT 48, *MTAC*, v. 2, 1946, p. 139–140.]
3. J. Glaisher, *Factor Table for the Fourth Million*, London, 1879, p. 48–52.

6[F, Z].—Edgar Karst, *Tables for converting 4 digit decimal fractions to periodic octal fractions.* 4 Tables, 13 handwritten sheets, 10.8 cm., deposited in the UMT File.

The author lists the complete period of 4, 20, 100, and 500 octal digits for the decimal fractions .2, .04, .008, and .0016. For any other fraction, the period is a cyclic permutation of one of these periods and the proper starting place is tabulated. If the fraction is not a multiple of 5^{-4}, determine the first digit or two in the usual way (multiply by 8) and the table may now be consulted for the periodic digits. The tables are arranged for the convenience of the table maker, rather than the user. A complete description is:

Table V—i

$5^{-i}k$ $\quad \mathrm{Ind}_8 k \quad n \quad 8^n 5^{-i} \pmod 8$

where $k = 1(1)5^i$ $(k, 5) = 1$ $n = 1(1)4 \cdot 5^{i-1}$ $i = 4(1)1$ and indices are with respect to 5^i.

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7[I].—Francis L. Miksa, *Stirling numbers of the first kind.* 27 leaves reproduced from typewritten manuscript on deposit in the UMT File.

After completing a table [4] of Stirling numbers of the second kind, ${}_rS_n$, for $n = 1(1)(50)$ the writer decided to prepare a companion table of Stirling numbers of the first kind, $S(r, n)$, computed for the same range.

Both kinds of Stirling numbers arise in formulating relationships between algebraic factorials and powers. For the Stirling I numbers we need only to express the factorials by means of power series.

It is customary to define the coefficients in these power series as the *Stirling numbers of the first kind*, and this of course includes the algebraic sign. For our purpose of tabulation, however, it is more convenient to consider only the *absolute value* of the coefficients.

The table was computed by means of the recurrence relation

$$S(r, n + 1) = n \cdot S(r - 1, n) + S(r, n)$$

and was checked by means of

$$\sum_{r=1}^{n} S(r, n) = n!.$$

Comparison with the table of *Exact Values of the First 200 Factorials* [1] showed that the agreement was complete.

The most extensive earlier table known to the writer is that published during 1900 by J. W. L. Glaisher [2], where they are listed up to $n = 23$ in our notation.

A more detailed treatment of applications of the Stirling I numbers appears in C. JORDAN [3].

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1. H. S. UHLER, *Exact value of the first 200 factorials*, New Haven, Conn., *MTAC*, v. 1, RMT 158, 1943–45, p. 312.
2. J. W. L. GLAISHER, "Congruences relating to the sums of products of the first n numbers and to other sums of products," *Quart. Jn. Math.*, v. 31, 1900, p. 1–35.
3. C. JORDAN, *Calculus of Finite Differences*, Chelsea Pub. Co., New York, 1947, p. 142–168.
4. FRANCIS L. MIKSA, *Stirling Numbers of the Second Kind*. Deposited in *MTAC*, UMT FILE. See Review 85, *MTAC*, v. 9, 1955, p. 198.

8[I].—KUO-CHU HO, "Double interpolation formulae and partial derivatives in terms of finite differences," *MTAC*, v. 9, 1955, p. 52–62.

Expressions for partial derivatives of $h^\beta k^{\alpha-\beta}(\partial^\alpha f/\partial x^\beta \partial y^{\alpha-\beta})$ where $\alpha = 1(1)4$, $\beta = 0(1)\alpha$ are given in terms of the differences of f at intervals h in x, k in y. The expressions given include the fourth order terms. These are given in various forms suitable for use at the edges and corners of a table as well as in the center. The corresponding Lagrangian expressions are given for $\alpha = 1(1)3$.

J. T.

9[I].—H. E. SALZER, "Osculatory quadrature formulas," *J. Math. Physics*, 34, 1955, p. 103–112.

Quadrature formulas such as

$$I(r, s) = \int_{x_0+rh}^{x_0+sh} f(x)dx = h \sum A_i f_i + h^2 \sum B_i f_i' + Rh^{2n+1} f^{(2n)}(\theta)$$

can be found by integration of osculatory interpolation formulas. The exact values of the A_i, B_i, R, expressed as rational fractions in their lowest terms, are given in the case of n points for $n = 2(1)7$. For each n the appropriate coefficients are given for all possible sub-intervals: thus for $n = 4$, with f and f' given at points $x_i = x_0 + ih$, $i = -1, 0, 1, 2$, we find expressions for the integrals $I(r, s)$ for the following values of (r, s): $(-1, 0)$, $(0, 1)$, $(1, 2)$; $(-1, 1)$, $(0, 2)$; $(-1, 2)$ in terms of *all* the data.

J. T.

10[I].—T. N. E. GREVILLE & H. VAUGHAN, "Polynomial interpolation in terms of symbolic operators," *Trans. Soc. Actuar.*, v. 6, 1954, p. 413–476.

Table 6, p. 450–451, gives the exact values of the basic function $L(x) = L(-x)$ of the following continuous operators: M^6, μM^5, M^4, $\delta^2 M^4$, $\delta^4 M^4$, μM^3, $\mu\delta^2 M^3$, $\delta^4 M^2$, M^5, μM^4, M^3, $\delta^2 M^3$, for $x = 0(.1)4$ or less (usually around 3), up to the value of x where $L(x) = 0$. The basic function $L(t)$ for the continuous operator K is defined by $K = \int_{-\infty}^{\infty} L(t)E^{-t}dt$ where E denotes the displacement operator of the calculus of finite differences. The operator K is also called the *characteristic operator* for the interpolation formula of which $L(t)$ is the basic function, and it is analogous to the graduation operator which the authors have associated with a discrete interpolation formula.

Preceding Table 6, on p. 443–449 are the following collections of formulas:

Table 1, p. 443–445, *Characteristic Operators of Certain Published Interpolation Formulas*, giving name or originator of formula, place and date of publication, and characteristic operators.

Table 2, p. 446, gives the symbolic expansions of certain operators, namely, combinations or separate powers of M, μ, and δ in terms of $D \equiv \frac{d}{dx}$. ($\delta = E^{\frac{1}{2}} - E^{-\frac{1}{2}}$, $\mu = \frac{1}{2}(E^{\frac{1}{2}} + E^{-\frac{1}{2}})$, and M is a new operator defined by $Mf(x) = \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x + t)dt$.)

Table 3, p. 446, Traces of powers M^k and mean powers μM^k, $k = 1(1)8$, in terms of even powers of δ, where the author defines the trace of a continuous operator K with basic function $L(t)$ as the discrete operator $t(K) = \sum_{-\infty}^{\infty} L(n)E^{-n}$.

Table 4, p. 447, Powers and mean powers of M, up to the 10th, in terms of special operators for both endpoint and midpoint formulas.

Table 5, p. 448–449, Various special operators for endpoint, midpoint, or mixed endpoint-midpoint formulas in terms of M, δ, μ, and D.

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11[K].—THE RAND CORP., *One Million Random Digits and 100,000 Normal Deviates*, The Free Press, Glencoe, Illinois, 1955, 26.5 cm., xxv + 200 p. $10.00. The tables are also available on punched cards from The RAND Corporation, 1700 Main St., Santa Monica, Calif. $25.00 an order +$3.00 per 1000 cards.

This book contains one million random decimal digits and 100,000 normal deviates, computed from the random digits to 3D. This succeeds [1] as the largest table of random numbers known to be available. The book also contains introductory material describing production of the digits, tests of the digits, production and tests of the normal deviates, and use of the tables.

Random numbers have long been used in various ways for the simulation of random or imponderable effects in various calculations. In recent years a growing popularity of the so-called Monte Carlo type of calculation in various shielding problems in physics and other problems of mathematics and applied mathematics has led to a greatly increased demand for random numbers; some of this recent work is described in [2], [3], [4]. The increasing power of modern computing equipment has also led to higher demands for random numbers in their more classical uses. The present publication was prepared in order to meet these increased demands. The largest earlier publication [1] is reported to have been completely inadequate in size for the needs of The RAND Corporation.

Generally speaking, workers have been content with the generation of numbers conforming with frequency distribution requirements and believed to be essentially independent of the process being simulated in their method of production. Many systematic ways of generating such numbers have been proposed for digital computers; references to some of these are contained in [6], and other schemes may be found in [7]. The present numbers were not generated by any known formula, but they were rather generated by a process which (if current physical

theory is correct) was completely random, yielding output which could not have been predicted with any amount of knowledge available before the generation of the numbers. The digits were prepared by driving electronic counters by electronically generated noises. (Another random method of generating binary digits is described earlier in this issue [11].)

The random digits were subjected to numerous tests described in the introductory material. The originally generated digits showed unacceptable biases in their distribution, and a new table was generated systematically by adding pairs of digits modulo 10 to improve (that is, to flatten) the distribution. This was successful. Standard tests of frequency distributions, poker hands, frequencies of digraphs (ordered pairs of digits), and lengths of runs of a single digit were carried out, presumably on the digits as they were punched on the IBM cards.

The printed version was prepared automatically from these IBM cards, and it is unlikely to differ from the punched card version in more than a few digits.

All the standard tests as reported were passed adequately by the digits. However, one test was improperly analyzed, and the authors have requested that the following correction be made in the text. It is believed that the desirability of some such modification was brought to the Corporation's attention by I. J. GOOD; it is based upon his analysis of a serial test [8].

Replace the paragraph beginning at the bottom of page xv by the following:

Table 5 can be tested by a criterion originally due to Kendall and Smith and revised by Good. Assuming all pairs equally likely we get a normalized sum of squared deviations of 107.8. However, this statistic does not have a χ^2-distribution. On the other hand, it is the sum of the error variation and twice the row (or column) variation, where under the assumption of perfect randomness, the error variation is asymptotically distributed like χ^2 with 81 degrees of freedom. We take the error variation as our test criterion. This gives a χ^2 of $107.8 - 2(7.56) = 92.7$, which is about the 0.18 level for 81 degrees of freedom.

The modification above cannot be interpreted as challenging the randomness of the digits.

It seemed conceivable that the standard tests reported as having been made in the introduction to the book would have failed to detect variable instability of counters (which, in any event, would be unlikely to have a bad effect on the randomness of the digits after the smoothing transformation made above). The reviewer did test a hypothesis that such instability occurred and led to excessive lengths of coupon collector's sequences [9]. Accordingly the digits were transformed from decimally punched card form to binary coded decimal form suitable for introduction to the SWAC computing machine located at the University of California at Los Angeles. The coupon collector's test described by GREENWOOD was run by the reviewer with the help of F. H. HOLLANDER, J. L. SELFRIDGE, and DAVID A. POPE. The longest single sequence which was required in order to provide at least one of each of the 10 digits had length 115 digits; the total number of sequences tested before the digits were exhausted was 34,248; the longest sequence of length 115 was not inordinately long. Based on the total number of sequences observed and the probabilities reported by Greenwood [9], the following comparison is possible between the numbers of sequences observed of various lengths and the numbers expected.

Length of Run i	Number Observed ν_i	Number Expected μp_i
10	14	12.428
11	63	55.926
12	152	143.542
13	284	276.832
14	462	446.786
15	640	638.190
16	813	834.267
17	1017	1020.064
18	1264	1184.234
19	1342	1319.539
20	1415	1422.517
21	1480	1492.752
22	1528	1532.020
23	1515	1543.501
24	1576	1531.128
25	1488	1499.087
26	1431	1451.483
27	1375	1392.115
28	1299	1324.359
29	1189	1251.116
30	1269	1174.806
31	1136	1097.397
32	977	1020.446
33	972	945.150
34	839	872.401
35	792	802.832
36	689	736.868
37	695	674.765
38	618	616.647
39	554	562.532
40	494	512.363
41	490	466.026
42	408	423.364
43	386	384.196
44	357	348.322
45	338	315.534
46	304	285.622
47	265	258.376
48	235	233.594
49	196	211.081
50	196	190.651
51	188	172.129
52	148	155.352
53	139	140.166

Length of Run i	Number Observed ν_i	Number Expected μp_i
54	114	126.428
55	117	114.009
56	104	102.788
57	84	92.652
58	74	83.502
59	70	75.243
60	67	67.793
61	53	61.072
62	62	55.012
63	53	49.549
64	42	44.624
65	47	40.186
66	37	36.186
67	22	32.583
68	41	29.337
69	21	26.413
70	28	23.780
71	20	21.408
72	15	19.273
73	18	17.349
74	14	15.618
75	11	14.059
76 and more	102	126.631

The reviewer cannot find any significant indication in these data to support a hypothesis that the digits are not random. A calculation of χ^2 based on the assumption that 34,248 samples were divided into 67 categories with 66 degrees of freedom gave a value of χ^2 about 0.135σ below the expected mean value. More detailed results of this coupon collector's test may be obtained by addressing the reviewer.

The normal deviates were computed from the random digits according to a standard procedure, see [6]. In particular, if D is a five-digit number taken from the table of random digits and read as an integer, then the numbers listed are values of x solving the equation

$$(D + 0.5)10^{-5} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-t^2/2}dt.$$

For the right hand number of this equation the National Bureau of Standards tables of the normal distribution [10] were used.

The printing is satisfactory. Reproduction was by photographic offset printing from pages printed by an International Business Machine Model 856 Cardatype. Each page of the table of random digits contains fifty lines; each line contains a serial number and ten groups of five decimal digits. The digits are easily legible. Each page of the table of normal deviates contains fifty lines; each line contains

a serial number and ten deviates. The material was proofread only on a sample basis, but the probability of serious error in printing seems low.

C. B. T.

1. M. G. KENDALL & B. B. SMITH, *Random Sampling Numbers*, Cambridge University Press, 1939.
2. NATIONAL BUREAU OF STANDARDS, *The Monte Carlo Method* (Proceedings of a Symposium held in 1949), NBS AMS 12, U. S. Gov. Printing Office, Washington, D. C., 1951.
3. J. H. CURTISS, "Sampling methods applied to differential and difference equations," *Seminar on Scientific Computation*, International Business Machines Corp., New York, 1949.
4. H. KAHN, Applications of Monte Carlo, The RAND Corp. (to be published).
5. P. DAVIS & P. RABINOWITZ, "Some Monte Carlo experiments in computing multiple integrals," *MTAC* (this issue), v. X, 1956, p. 1.
6. D. L. JOHNSON, "Generating and testing pseudo random numbers on the IBM Type 701," *MTAC* (this issue), v. X, 1956, p. 8.
7. L. SACCO, *Manuel de Cryptographie*, Payot, Paris, 1951, Chap. XI, p. 74–119.
8. I. J. Good, "The serial test for sampling numbers and other tests for randomness," Camb. Phil. Soc., *Proc.*, v. 49, 1953, p. 276–284.
9. ROBERT E. GREENWOOD, "Coupon collector's test for random digits," *MTAC*, v. IX, 1955, p. 1–5. (Note also Corrigendum, *MTAC*, v. IX, 1955, p. 229.)
10. *Tables of Probability Functions*, 2nd ed., U. S. Gov. Printing Office, Washington, D. C., 1948.
11. Z. PAWLAK, "Flip-flop as Generator of Random Binary Digits," *MTAC* (this issue), v. X, 1956, p. 28.

12[K].—NBS Applied Mathematics Series, No. 44, *Table of Salvo Kill Probabilities for Square Targets*, U. S. Gov. Printing Office, Washington, D. C., 1954, ix + 33 p., 26 cm. Price $0.30.

Let $\{x, y, \sigma\}$ denote the circular normal distribution centered at (x, y) with standard deviation σ in each direction. A salvo of N bombs is "centered" at (ξ, η), which has the distribution $\{0, y_0, \sigma_A\}$. The bombs of the salvo are independently distributed according to $\{\xi, \eta, \sigma_B\}$. If a bomb hits the square target ($x^2 \leq 1$, $y^2 \leq 1$), there is chance P_K of a kill. Assuming the bombs act independently, the chance P_{SK} that the salvo kills the target is computed for $P_K = .1, .4, .7, 1$; $y_0 = 0, 1, 2, 4, 7, 11, 16, 22$; $\sigma_A, \sigma_B = 1, 2, 4, 7, 11, 16, 22$; $N = 1, 5, 10, 25, 50, 100, 150, 200$. The entries are 4D with possible error of two in the last place. In an introduction, A. D. HESTENES warns the user against interpolating in this quintuple-entry table, except in the P_K direction for small N.

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13[K].—K. C. S. PILLAI & K. V. RAMACHANDRAN, "On the distribution of the ratio of the ith observation in an ordered sample from a normal population to an independent estimate of the standard deviation," *Ann. Math. Stat.*, v. 25, 1955, p. 565–572.

Tables to 2D are given for (1), the 95th percentile of distribution of $q_n = x_n/x$; (2), the 5th, and (3), the 95th percentiles of the distribution of $u_n = |x_n/s|$ where x_n is the largest of a sample of n observations from a normal population with zero mean and unit variance and s is independently distributed with ν degrees of freedom, for (1): $n = 1(1)8$: $\nu = 3(1)10(2)20, 24, 30, 40, 60, 120, \infty$; (2): $n = 1(1)10$: $\nu = 1(1)5(5)20, 24, 30, 40, 60, 120, \infty$; (3): $n = 1(1)8$: $\nu = 5(5)20, 24, 30, 40, 60, 120, \infty$, respectively.

A table used in the derivation of the above tables is given to 8S for coefficients $a_i^{(k)}$ which are defined by

$$\left(\int_{-\infty}^{x} \frac{e^{-t^2/2}}{\sqrt{2\pi}}\,dt\right)^k = e^{-kx^2/6}\left(\sum_{i=0}^{\infty} a_i^{(k)}\chi^i\right)$$

for $i = 0(1)30$; $k = 1(1)7$.

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14[K].—P. B. PATNAIK, "A test of significance of a difference between two sample proportions when the proportions are very small," *Sankhyā*, v. 14, 1954, p. 187–202.

The paper is concerned with the following problem: "if there is a random sample of N_1 individuals from a population, of which x_1 have the characteristic A, and a random sample of N_2 from a second population, of which x_2 have A, then it is desired to test whether the chance of possessing A is the same in the two populations." Consideration is restricted to cases in which Poisson distributions are acceptable approximations to the binomial distributions which give the probabilities for various values of X_1 and of X_2. The problem thus becomes one of testing, using one observation from each Poisson distribution, the hypothesis that the ratio of the parameters of the two Poisson distributions is a given constant. Two test procedures are proposed. Randomized tests are not considered. (A) It is pointed out that, if the conditional probabilities of falling in the critical region for fixed sum of the two observations is kept below some preassigned value, the size of the critical region is, for small sum of the parameters, very much below this preassigned value. One of the many possible methods of interpolating is proposed to keep conditional probabilities near, but not necessarily below, the preassigned number and to make the size of the critical region near this number. Some tables illustrate the effects on conditional and unconditional sizes of critical regions. (B) The idea of a "test with minimal bias" is introduced. The power curve must be at or below the nominal size at the parameter value for the hypothesis under test, derivatives at this point must satisfy certain conditions, and among tests satisfying these conditions, that one is chosen which minimizes the length of the interval of parameter values for which the power curve is below the nominal size. The requirement of minimal bias is imposed on the conditional regions, making the test that of a binomial variate. Critical values for a binomial variate are given (in Table 6) for α(nominal size) $\equiv 0.05, 0.10$; ρ(parameter of the binomial) $= \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}$, n(index at the binomial) $= 6(1)25$. Several typographical errors in titles are obvious.

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15[K].—BENJAMIN EPSTEIN, "Truncated life tests in the exponential case," *Ann. Math. Stat.*, v. 25, 1954, p. 555–564.

With n items on a life test, it is decided in advance that the experiment is terminated either at a fixed truncation time T_U or at the time $X(r_0, n)$ necessary

for the r_0th failure, whichever is smaller (stopping rule), and it is assumed that the distribution of lives $X (X \geq U)$ is exponential with unknown mean θ. If $X(r_0, n) < T_0 (> T_0)$ the null hypothesis $\theta = \theta_0$ is rejected (accepted). Two cases are considered, non-replacement and replacement from the same population. Formulae are derived for $E_\theta(r)$, the expected number of observations, and $E_\theta(T)$, the expected waiting time necessary to reach a decision. In the replacement case they are related simply by $E_\theta(T) = \frac{\theta}{n} E_\theta(r)$. The probability $L(\theta)$ of accepting $\theta = \theta_0$ where θ is true is calculated as the probability of reaching a decision requiring up to r failures. However, in EPSTEIN's formulae (2), (13), (21), the sign = should read $\geq$.

Table 1 gives the values of r and $\chi^2_{1-\alpha}(2r)/(2r)$ such that the test based on using $\hat{\theta}_{r,n} > C = \theta_0 \chi^2_{1-\alpha}(2r)/2r$ as acceptance region for $\theta = \theta_0$ will have the probabilities $L(\theta_0) = 1 - \alpha$ and $L(\theta_1) \geq \beta$ for the errors of the first type with $\alpha = .01, .05, .10$ and of the second type with $\beta = .01, .05, .10$ for $\theta_0/\theta_1 = \frac{3}{2}, 2, \frac{5}{2}, 3, 4, 5, 10$. (The heading of the last column should be $\beta = .10$ instead of $\beta = .01$.) Table 2 gives some integer values of $n = [\theta_0 \chi^2_{1-\alpha}(2r_0)/2T_0]$ to be used in truncated non-replacement procedures for $\alpha = .01, .05$; $\beta = .01, .05$ for $\theta_0/\theta_1 = 2, 3, 5$ and $\theta_0/T_0 = 3, 5, 10, 20$.

Excellent practical illustrations are given as solutions of problems of the following type: Find a truncated replacement plan for which $T_U = 500$ hours which will accept a lot with mean life $\theta = 10{,}000$ hours at least 95% of the time and reject a lot with mean life $\theta = 2000$ hours at least 95% of the time. Then $\alpha = \beta = .05$. The values $L(\theta)$, $E_\theta(T)$ and $E_\theta(r)$ are computed for both values of θ.

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16[K].—D. B. DUNCAN, "Multiple range and multiple F tests," *Biometrics*, v. 11, 1955, p. 1–42.

Let $q(p, n_2) = w/s$ where w is the range of p independent normal variables having the same mean and unit standard deviation, and $n_2 s^2$ is distributed independently of w as chi square with n_2 degrees of freedom.

Table II labeled "Significant studentized ranges for a 5% level new multiple range test" lists the $(.95)^{p-1}$ percentiles of the sampling distributions of $q(p, n_2)$ for $p = 2(1)10(2)20, 50, 100$ and $n_2 = 1(1)20(2)30, 40, 60, 100, \infty$. Table III lists the $(.99)^{p-1}$ percentiles for the same distributions. These tables are used in performing comparisons among the means of equal sized samples from p populations. Tables of the 95th and 99th (same percentiles for all p) percentiles of $q(p, n_2)$ are given in Biometrika Tables [1].

Care should be taken not to confuse the 5% level heading in the table (what the author calls 95% two mean protection level) with 5% level of significance in an analysis of variance test since they actually refer to different probabilities. If an analysis of variance test is to be performed at the 5% level to test the hypothesis that the p populations have equal means then, if the p means are equal,

there is a 5% probability that the data will be such that the hypothesis is rejected. For the tests suggested in this paper there is a 5% least upper bound to the probability that any one pre-specified pair of populations having equal means will be judged to have unequal means. Using the suggested test for a case where all p populations have equal means the probability that at least one pair will be declared significantly different is $1 - (.95)^{p-1}$.

This is an attempt to make the suggested tests more comparable in overall level of significance to the case in analysis of variance where single degrees of freedom are used, and repeated tests at the 5% level are used.

Tables I and IV describe an example.

Table V classifies several discussed test procedures according to the measure of variability used among the means (range or variance), whether having other observed means between two specified means changes the difference labeled significant, and whether these differences depend on a fixed level α or a level which changes with the number of means between the two specified ones.

Table VI lists for comparison some "significant ranges for 5% level tests" of four discussed tests. Table VII compares the power of two classification of tests ($p = 20$, 5% level) to recognize a difference in the means of two pre-chosen populations.

Due to the fact that "5% level" means different things for the different tests Tables VI and VII are likely to give a somewhat slanted comparison of the various discussed tests.

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1. E. S. PEARSON & H. O. HARTLEY, *Biometrika Tables for Statisticians*, v. I, Cambridge, 1954.

17[K].—E. C. FIELLER & H. O. HARTLEY, "Sampling with control variables," *Biometrika*, v. 41, 1954, p. 494–501.

Let x and y be jointly distributed, and let $f(x, \theta)$, $g(y, \phi)$ be the respective marginal distributions. Information about the control variable x is available, *e.g.*, $f(x, \theta)$ may be completely known or θ may be known. The authors are concerned with using this information in the estimation of ϕ for the case where $X_1, \cdots, X_5$ are observations from an $N(\mu, 1)$ population, $x = \sum(X_i - \bar{X})^2$, $y = X_{\max} - X_{\min}$; the distribution of y is assumed unknown and that of x known. In the determination of the variance of the proposed estimate, the expectation of the reciprocal of a binomial variate, excluding the zero class, is needed.

Values of (1) $E'(np, n) = \sum_{r=1}^{n} \frac{1}{r}\binom{n}{r} p^r q^{n-r}$ for $m \equiv np = 1(1)10$, $n = 25$, 50, ∞ are computed directly. For $m = 12(2)20$, $n = 25$, 50, 100,

$$(2)\quad E'(m, n) \simeq \frac{1}{m+p} + \frac{1}{(m+p)(m+2p)} + \frac{2!}{(m+p)(m+2p)(m+3p)} + \cdots$$

is used; for $m = 25(5)45$, $n = 50$, 100 a control is used with (2). $E'(m, \infty)$ may be calculated from $E'(m, \infty) = e^{-m}\{\mathrm{Ei}(m) - \log_e m - \gamma\}$ using available tables

[1, 2]. For large np, the approximation $E' \simeq \frac{1}{m-q} + \frac{q}{(m-q)^3} + \frac{q(q+1)}{(m-q)^4}$ is is given. Values of (1) for $m = 1(1)10$ are given to 4D, all others to 5D.

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1. BAAS *Math. Tables*, v. I, London, 1931.
2. E. JAHNKE & F. EMDE, *Tables of Higher Functions*, 5th ed. Leipzig: 1952

18[K].—P. N. SOMERVILLE, "Some problems of optimum sampling," *Biometrika*, v. 41, 1954, p. 420–429.

Let

$$f(y) \equiv f(y_1, \cdots, y_k) = (2\pi)^{-k/2}|\Sigma|^{-k/2} \exp -\tfrac{1}{2} \operatorname{tr} y\Sigma^{-1}y',$$

where $y = (y_1, \cdots, y_k)$, $\Sigma = (\sigma_{ij})$, $\sigma_{ii} = 1$, $\sigma_{ij} = \frac{1}{2}$ and let $F_k(x) = \int_R f(y)\pi dy_i$, where R: $-\infty < y_i < x$, $i = 1, \cdots, k$. In Table 3.1 the author computes $F_k(x)$, $x = 0(.1)2(.5)3$ to 5D for $k = 1, 2$; 4D for $k = 3, 4$; 3D for $k = 5$. $F_1(x)$ is available in a number of tables. $F_2(x)$ was computed using Table VIII of part II of the Pearson Tables [1], and $F_1(x)$. For $F_k(x)$, $k = 3, 4, 5$, an expansion of $f(y)$ in terms of Hermite polynomials was used [2].

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1. K. PEARSON, *Tables for Statisticians and Biometricians*, London, 1931, part II.
2. W. F. KIBBLE, "An extension of a theorem of MEHLER's on Hermite polynomials." Camb. Phil. Soc., *Proc.*, v. 41, 1945, p. 12–15. See p. 15.

19[K].—H. F. DODGE, "Skip-lot sampling plan," *Industrial Quality Control*, v. XI, no. 5, 1955, p. 3–5.

For derivations, formulae, and basic procedure, see the author's earlier paper [1]. The procedure developed there is as follows: (1) inspect 100% of units consecutively as produced until i successive good units are found (each unit is good or bad), (2) thereupon, inspect only an unbiased proportion f, (3) if in the proportion f a bad unit is found, revert to 100% inspection until i successive good units are found, etc., (4) replace bad by good units. For given f and i, the average outgoing proportion defective (AOQ) as a function of the incoming proportion defective (p), and (consequently) the maximum AOQ ($AOQL$) are determined.

In the current paper, the series of units are replaced by series of lots, inspection of a unit by analysis of a sample, bad unit by bad lot (lot whose sample fails to meet a standard). Corresponding changes of meanings of p, AOQ, and $AOQL$ to apply to lots rather than units.

Figure 1 gives the $AOQL = .01(.01).06, .08, .10$ as a function of $i = 1(1)50$, $(2)100$ and $f = .01(.001).05(.002).10(.01).50$. Figure 1 is essentially an abridgement of Figure 3 of his 1943 paper cited above (p. 272) though the $AOQL$ curves are now fully shown for low i.

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1. H. F. DODGE, "A sampling plan for continuous production," *Ann. Math. Stat.*, v. 14, 1943, p. 264–279.

20[K].—J. AITCHISON & J. A. C. BROWN, "An estimation problem in quantitative assay," *Biometrika*, v. 41, 1954, p. 338–343.

A popular model for the quantitative response u to a stimulus of concentration x is: $u = HP(\alpha + \beta x) + \epsilon$, where H is the maximum expected response; $P(\alpha + \beta x)$ is the normal cumulative based on a linear function of x; and ϵ is $N(0, \sigma^2)$, independent of x. The authors consider a model of the form:

$$\ln u = v = \ln H + \ln [P(\alpha + \beta x)] + \epsilon = K + \ln P + \epsilon,$$

i.e., Var (u) is proportional to $[E(u)]^2$. Iterative equations, familiar to probit users, are set up to estimate α, β, and K.

A table of working probits, to facilitate the solution of these equations, is given for initial guessed probits, $Y = a_0 + b_0 x = 1.0(0.1)9.0$. This table contains values of the minimum working probit to 4D; of the auxiliary variable, P/Z to 4D or 5S; and of the weighting factor, Z^2/P^2 to 4D. The working probit for this model is $y = \left(Y - \frac{P}{Z} \ln P \right) + (v - K) \frac{P}{Z}$.

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21[K].—A. R. JONCKHEERE, "A distribution-free K-sample test against ordered alternatives," *Biometrika*, v. 41, 1954, p. 133–145.

Let X_i, $i = 1, 2, \cdots, k$, be independent random variables with the continuous cumulative probability functions $F_i(x)$, and let X_{ij}, $j = 1, 2, \cdots, m_i$, be samples of size m_i of the X_i. To test the hypothesis that $X_1, \cdots, X_k$ have the same distribution (H_0): $F_1(x) = F_2(x) = \cdots = F_k(x)$ against the alternative that these random variables are stochastically increasing (A): $F_1(x) > F_2(x) > \cdots > F_k(x)$ (on p. 134 the sense of these inequalities is inverted, as it is in the footnote on p. 135), the author considers the following statistic: Let p_{ij} = number of pairs X_{ir}, X_{js} such that $X_{ir} < X_{js}$, among all possible $m_i m_j$ pairs, for given $i = 1, \cdots, k-1$; $j = i + 1, \cdots, k$. The test statistic is $S = 2 \sum_{i=j}^{k-1} \sum_{j=i+1}^{k} p_{ij} - \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} m_i m_j$, identical with a statistic considered by M. G. KENDALL [1] and closely related to the rank correlation coefficient.

In his study of the probability distribution of S under (H_0) the author obtains the first four cumulants, studies their extreme values, and obtains asymptotic distributions under several assumptions. He finds in particular that if at least two of the sample sizes m_i tend to infinity, then the limit distribution of S is normal, and he proposes a somewhat better approximation by STUDENT's distribution. Finally, the exact distribution for small samples is given.

Table 3 gives Pr $\{S \geq S_0\}$ to 3 or 4S for k samples, each of size m, covering the range: $k = 3$ and $m = 2, 3, 4, 5$; $k = 4$ and $m = 2, 3, 4$; $k = 5$ and $m = 2, 3$; $k = 6$ and $m = 2$; $S_0 = 0(2) \cdots$ until Pr $\{S \geq S_0\}$ becomes negligible.

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1. M. G. KENDALL, *Rank Correlation Methods*, London, 1948.

22[K].—R. A. BRADLEY, "Rank analysis of incomplete block designs II. Additional tables for the method of paired comparisons," *Biometrika*, v. 41, 1954, p. 502–537.

The present article extends the 1952 tables of BRADLEY and TERRY [1] to include 7 and 8 repetitions for 4 treatments ($t = 4; n = 7, 8$) and 1 to 5 repetitions for 5 treatments ($t = 5; n = 1, 2, 3, 4, 5$). The problem considered is to estimate true treatment ratings ($\pi_1, \cdots, \pi_t; \Sigma\pi_i = 1$) and test the hypothesis H_0: all $\pi_i = 1/t$. The tables give, for all combinations of total rank sums, estimates of the π_i to 2D, the value of the testing statistic, B_1 to 3D, and the significance probability for B_1 to 4D. Only two treatments are tested at a time, giving $\binom{t}{2}$ pairs per repetition. The favored treatment in each pair is given a rank of 2 and the other a rank of 1; hence, the total rank sum for any treatment can fall between $n(t-1)$ and $2n(t-1)$.

The author includes (i) a study of the usefulness of an approximate statistic, (ii) a discussion of various uses of B_1, and (iii) two errata for the original tables.

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1. R. A. BRADLEY & M. E. TERRY, "Rank analysis of incomplete block designs. I. The method of paired comparisons," *Biometrika*, v. 39, 1952, p. 324–345 [*MTAC*, v. 8, 1954, p. 17].

23[K].—EDWARD WALTER, "Über die Ausnutzung der Irrtumswahrscheinlichkeit," *Mitteilungsblatt für Math. Stat.*, v. 6, 1954, p. 170–179.

Tests of significance based on statistics which can assume only a finite number of different values usually do not attain their asserted level of significance. It is always possible to modify the test (by introducing a randomized test) in order to attain the asserted level of significance. The author is interested, however, in a modification based on the observations. He considers the sign test for testing the null hypothesis that the distribution function is symmetric about zero. The modified sign test is defined as follows. Let y equal the number of negative observations in the sample of n; let y' be the rank of the negative observation with largest absolute value among the absolute values of all the observations; let y'' be the rank of the negative observation with second largest absolute value among the absolute values of all the observations; etc. The null hypothesis is to be rejected if $y < y_0$, or $y = y_0$ and $y' < y_0$, or $y = y_0$, $y' = y_0'$ and $y'' < y_0''$, etc. In table 1 values of $y_0, y_0', y_0'', \cdots$ are given for $n = 5(1)25$, $\alpha = .01, .05$, and for one- and two-sided tests. The two-sided test is the same as the one-sided test except that one considers either the number of positive observations or the number of negative observations, according to which is the smaller.

In table 2, empirical results are tabulated to compare the power of the sign test, modified sign test, and the t-test for $\alpha = .01, .05$ in the case where the distribution is normal with variance 1 and mean $\mu = .6, 1, 1.4$.

In table 3, the effective levels of significance of the sign test and modified sign test are given for $n = 5(1)30$.

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24[K].—SHOZO SHIMADA, "Power of *R*-charts," *Reports of Statistical Application Research*, Union of Japanese Scientists and Engineers, v. 3, 1954, p. 70–74.

Let x_1, x_2, x_3, x_4 be independent and normally distributed random variables with means m_1, m_2, m_3, m_4 and common variance σ^2. The range R is defined as the difference between the largest and smallest value among the four successive observations.

If $m_1 = m_2 = m_3 = m_4$, then the range provides an estimate of σ. On the other hand, if not all the m_i's are equal then the range may include the effect of the variability of the m_i's in its estimation of σ.

The author finds the distribution of R based on four observations when the m_i's are not all equal. He then plots the power curves of the range chart for three special cases, approximating a two fold integral by using "Circular Probability Paper" [1].

He considers the following three cases:

Case 1. $m_i = \alpha + (i - 1)\eta$ where α and η are constants.

Case 2. $m_i = m_j = m_k + \delta = m_l + \delta$ (i, j, k, or l represents one of four integers 1, 2, 3, 4, with no two equal).

Case 3. $m_i = m_j = m_k = m_l + \delta$.

Table 1 gives the probability that a point plotted on a range chart falls outside a specified control limit for the three different cases. Table 2 gives the probability that x_i and x_j take on the smallest and largest value for Case 1 where $\eta = .05$.

Some typographical errors are listed below:

1. Fig. 2 Case 1 should be Fig. 3 Case 2.
2. Fig. 3 Case 2 should be Fig. 2 Case 1.
3. In Table 2, the headings Max. and Min. should be interchanged.

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1. F. C. LEONE & C. W. TOPP, "Circular probability paper," *Industrial Quality Control*, v. 9, 1952, p. 10–16.

25[K].—C. T. FAN, "Note on the construction of an item analysis table for the high-low-27-per-cent method," *Psychometrika*, v. 19, 1954, p. 231–237.

The table whose construction is described in this article provides a means of translating the observed proportions of success in the high and low 27 percent groups (denoted by p_H and p_L) into measures of item difficulty (denoted by p and Δ) and of item discrimination (denoted by r). Values are tabled for the difficulty index, p, and the discrimination index, r, as functions of p_H and p_L. These results are based on KARL PEARSON's tables of the normal bivariate surface. A second item difficulty index, Δ, can be determined from p and is expressed in terms of a normal curve deviate with mean 13 and standard deviation 4. The final complete table is not given in this paper but has been published by the Educational Testing Service, Princeton, N. J. However, a graphical version of the final results is given as figure 2 in this paper. Preliminary computations in the table construction are presented in table 3, which furnishes p_H and p_L values to 4D for $.5000 \leq p \leq .9713$ and $r = .05(.05)1.00$. Figure 2 furnishes a simplified

version of the final chart for use in graphical estimation of p and r for given values of p_H and p_L. This graph contains a square grid for $p, r = .00(.10)1.00$ and curves of corresponding p_H and p_L values for $p_H p_H = .10(.10).90, .99$ and $p_L = .01, .10(.10).90$.

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26[K].—W. T. SHARP, J. M. KENNEDY, B. J. SEARS, & M. G. HOYLE, *Tables of Coefficients for Angular Distribution Analysis*, Atomic Energy of Canada, Ltd., Chalk River, Ontario, A.E.C.L. Report No. 97, 1954, xxxix + 38 p., $2.00

The functions of angular momentum quantum numbers introduced by Racah have related closely the studies of the various correlations in angle of particles and radiation absorbed or emitted by nuclei. In the Chalk River tables the authors have compiled and calculated values of the Racah W-function, the associated coefficients Z and Z_1, and the 9-j symbol, X, for ranges of parameters which occur in low energy nuclear reactions.

The coefficients $W(ljl'j'; sk)$ and $Z(ljl'j'; sk)$ are in the main taken from the tables of BIEDENHARN [1] and OBI *et al* [2]. The range of s, the channel spin parameter, is extended to 7/2; a few values for $s = 4$ not given by Obi are newly computed. Except in a few cases, $j' = j$; the range of j is $0(\frac{1}{2})5$. l and l' are restricted to integers less than 4, whether or not the selection rules give non-zero coefficients for higher integers. A table of $W(jj_1jj_1; Lk)$ is provided for $j, j_1 = \frac{1}{2}(1)\frac{11}{2}$ and $L = 1, 2$.

The coefficients $Z_1(LjL'j'; Ik)$ are for $j' = j$ closely related to the $F_k(LIj)$ and $G_k(LL'Ij)$ of BIEDENHARN and ROSE [3]. The present tabulation, for $L, L' = 1, 2, 3$, $j' = j = 0(\frac{1}{2})6$, $I = 0(\frac{1}{2})6$, extends previous tables [4] to the parameter values $I = 5, 6$ and $j = 6$.

In all cases k is restricted to the even integers.

New tables of the coefficient $X\begin{vmatrix} abc \\ def \\ ghk \end{vmatrix}$ are given for $a, d = 1, 2$, $e = b$, $f = c$, $b, c = 1(\frac{1}{2})5$ and $g, h, k = 0(2)8$. A second specialization is made to $k = 1$, $a = d = 1(\frac{1}{2})3$, $g = h = 2, 4$, with the other parameters in the range $1(\frac{1}{2})3$.

Finally, auxiliary tables are provided for the triangle coefficient $\Delta(a, b, c)$, and for the CLEBSCH-GORDON coefficients $(ll'00|k0)$, $l, l' = 0(1)6$, and $(LL' - 11|k0)$, $L, L' = 1, 2, 3$.

The square of a coefficient is in each case a rational fraction. The quantities tabulated are the (positive or negative) powers of the prime factors of each square, and the sign of the coefficient.

A useful introductory section lists many of the important properties of the coefficients and discusses their application in the theory of nuclear reactions, heavy particle-gamma ray correlation, gamma-gamma correlation, transformations between angular momentum coupling schemes, triple correlation, and transition probabilities for electromagnetic radiation.

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1. L. C. Biedenharn, Oak Ridge National Laboratory, Reports No. 1098 and 1501.
2. S. Obi *et al.*, Annals of the Tokyo Astronomical Observatory III, 1953, p. 89.
3. L. C. Biedenharn & M. E. Rose, *Rev. Mod. Phys.*, v. 25, 1953, p. 729.
4. W. T. Sharp, Gove, & Taul, Chalk River Reports. PD. no. 254, unpublished.

27[K].—J. M. Kennedy, B. J. Sears, & W. T. Sharp, *Tables of X Coefficients*, Atomic Energy of Canada, Ltd., Chalk River, Ontario, 1954, A.E.C.L. Report No. 106, ix + 16 p., $1.00.

This work extends the authors' table of X coefficients in "Tables of Coefficients for Angular Distribution Analysis" [1]. The following tables are provided:

1. $X^2\begin{vmatrix} abc \\ abc \\ ghk \end{vmatrix}$ for $a = 1, 2, b, c = 1(1)5$. Here and throughout g, h, k run over all even integers compatible with the selection rules for X.

2. $X^2\begin{vmatrix} 1bc \\ 2bc \\ ghk \end{vmatrix}$ for $b, c = 1(1)5$.

3. $X^2\begin{vmatrix} abc \\ abc \\ ghk \end{vmatrix}$ for $a = 1, 2, b, c = \frac{1}{2}(1)\frac{9}{2}$; and for $a = 3, 4, b = \frac{1}{2}(1)\frac{9}{2}, c = \frac{1}{2}(1)\frac{9}{2}$.

4. $X^2\begin{vmatrix} 1bc \\ 2bc \\ ghk \end{vmatrix}$ for $b, c = \frac{1}{2}(1)\frac{9}{2}$.

5. $X^2\begin{vmatrix} a\frac{1}{2}c \\ abc \\ ghk \end{vmatrix}$ for $b = \frac{3}{2}, \frac{5}{2}, a = 1(1)4, c = \frac{1}{2}(1)\frac{9}{2}$.

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1. W. T. Sharp, J. M. Kennedy, B. J. Sears, & M. G. Hoyle, A.E.C.L. Report No. 97, 1954, Chalk River, Ont. See preceding review.

28[L].—N. W. McLachlan, *Bessel Functions for Engineers*, Oxford University Press, New York, 1955. Second edition. xii + 239 pages, 24 cm. Price, $5.60.

This second edition is considerably larger than the first edition, which appeared about twenty years ago. It still is a satisfactory text on the use of Bessel functions in classical physics and engineering problems. A little more has been added on the mathematical properties of the functions and a number of different applications have been added. The tone is strictly practical throughout, fundamental equations are quoted, discussed and applied, but not always proved. The style is condensed but readable. Short five-place tables, included at the back of the book, are for $J_n(x)$ for $n = 0, 1, x = 0(0.1)16$; for $n = 2, 3, 4, x = 0(0.1)5$; for zeros of these J's; for $Y_n(x)$ and $H_n(x)$, $n = 0, 1, x = 0(0.1)16$; for I_0, I_1, K_0, K_1 for $x = 0(0.1)10$; and for similar ranges of Ber and Bei, Ker and Kei, and their derivatives.

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29[L].—MORIO ONOE, "Formulae and tables, the modified quotients of cylinder functions," Report of the Institute of Industrial Science, University of Tokyo, v. 4, 1955, No. 5 (Serial No. 32), 22 p.

$C_\nu(z)$ being any cylinder function [1], the author sets

$$\mathfrak{C}_\nu(z) = zC_{\nu-1}(z)/C_\nu(z), \quad \tilde{\mathfrak{C}}_\nu(z) = zC_{\nu+1}(z)/C_\nu(z),$$

and similarly for Bessel functions of the first, second, and third kinds. He gives a collection of formulas regarding these functions, and also numerical tables to 4 to 7S, each accompanied by graphs.

Table 1. $\mathfrak{J}_n(x)$ for $n = 1(1)5$, $x = 0(.1)5$ and the first zero and the first pole of $\mathfrak{J}_1(x)$ with corresponding values of $\mathfrak{J}_n$.

Table 2. $\mathfrak{J}_n(iy)$ for $n = 1(1)5$, $x = 0(.2)6$.

Table 3. $\mathfrak{Y}_n(x)$ for $n = 1(1)5$, $x = 0(.1)5$ with the first two zeros of $\mathfrak{Y}_1$ and the first zeros of $\mathfrak{Y}_2, \mathfrak{Y}_3, \mathfrak{Y}_4$.

Table 4. $\mathfrak{H}_n^{(1)}(iy)$, $n = 1(1)5$, $x = 0(.1)5$.

Table 5. $\mathfrak{J}_n(n)$, $\mathfrak{Y}_n(n)$ for $n = 1(1)50$.

Appendix. Some coefficients in power series expansions of $\mathfrak{J}_n$ and asymptotic expansions of $\mathfrak{H}_n^{(1)}$, $n = 1(1)5$.

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1. G. N. WATSON, *Bessel Functions*, Cambridge, 1922, p. 82.

30[L].—H. E. SALZER, "Complex zeros of the error function," *J. Franklin Institute*, 260, 1955, p. 209–211.

Let $f(z) = \int_0^z e^{-u^2}du$. The zeros z_n of $f(z)$, for which $\frac{1}{4}\pi < \arg z_n < \frac{1}{2}\pi$, and $0 < |z_1| < |z_2| < \cdots$ are tabulated as follows: $n = 1$ to 12D, $n = 2(1)10$ to 9D. This table extends and corrects one of T. LAIBLE (*Z. ang. Math. u. Phys.*, 1951, p. 484–486).

J. T.

31[L].—HAROLD OSTERBERG & GORDON L. WALKER, *Table of* $\int_0^z [J_1(x)/x]dx$, Research Center, American Optical Company, Southbridge, Massachusetts, Communication No. 1, September 1, 1955. 7 multilithed pages with cover. 28 cm., deposited in the UMT FILE.

This table lists the function $f(z) = \int_0^z x^{-1}J_1(x)dx$ for $z = .01(.01)3.85$ and $z = 4(1)25$ plus about seven special values of the argument. This is the function $g(z)$ of "A guide to tables of Bessel Functions," by R. C. ARCHIBALD, *MTAC*, v. 1, 1944, p. 247. The numbers shown are usually 7S, and accuracy is claimed through 5S or 6S.

For the early range, the calculations were carried out on an International Business Machines Card Programmed Calculator, and for the later range they were carried out on a desk machine, using tables of Bessel Functions already published.

A similar, less extensive table has been published by R. GANS, "Mikroskopische probleme," *Annalen der Physik*, s. 4, v. 78, 1925, p. 1–34. The authors

note that their function is equal to $\int_0^x J_0(x)dx - J_1(z)$. The function $\int_0^x J_0(x)dx$ has been tabulated to 10D by A. N. LOWAN and M. ABRAMOWITZ in "Tables of integrals of $J_0(x)$ and $Y_0(x)$," *J. Math. and Phys.*, v. 22, 1943, p. 2–12, and this table has been reprinted as AMS 37 by the National Bureau of Standards; its range is $z=0(.01)10$.

C. B. T.

32[L].—B. ZONDEK, "The values of $\Gamma(\frac{1}{3})$ and $\Gamma(\frac{2}{3})$ and their logarithms accurate to 28 decimals," *MTAC*, v. 9, 1955, p. 24–25.

TABLE ERRATA

Reviews in this issue mention errata in the following works:

THE RAND CORPORATION, *One Million Digits and 100,000 Normal Deviates*, Review 11, p. 39–43.

BENJAMIN EPSTEIN, "Truncated life tests in the exponential test," *Ann. Math. Stat.*, v. 25, 1954, p. 555–564, Review 15, p. 44–45.

R. A. BRADLEY & M. E. TERRY, "Rank analysis of incomplete block designs. I. The method of paired comparisons," *Biometrika*, v. 39, 1952, p. 324–345, [*MTAC*, v. 8, 1954, p. 17], Review 22, p. 49.

SHOZO SHIMADA, "Power of *R*-charts," *Reports of Statistical Application Research*, Union of Japanese Scientists and Engineers, v. 3, 1954, p. 70–74, Review 24, p. 50.

T. LAIBLE, "Höhenkarte des Fehler-integrals," *Z. ang. Math. u. Phys.*, 1951, p. 484–486, Review 30, p. 53.

247.—GIUSEPPE PALAMA & L. POLETTI, "Tavola dei numeri primi dell'intervallo 12 012 000–12 072 060," Unione Matematica Italiana, *Bollettino*, s. 3, v. 8, 1953, p. 52–58. (*MTAC*, v. 7, 1953, p. 173, Review 1101[F].)

The following errata have been found.

Entry	Division	Probably intended prime
12 019 307	277	—
12 020 023	1901	—
12 023 381	31	12 023 383
12 028 813	131	12 028 817
12 045 149	457	—
12 047 023	107	—
12 071 881	2081	—

In addition the following primes should be added to the list.

12 047 309
12 069 919

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Editor's note: Primality of each number listed above as prime has been verified on the SWAC computer by J. L. SELFRIDGE.

CORRIGENDA

Review **56[C, D, E, K, L, S]**.—CECIL HASTINGS, JR., JEANNE T. HAYWARD, & JAMES P. WONG, JR., "Approximations for Digital Computers," *MTAC*, v. 9, p. 122, line 6, *for* k_1 *read* K_1 in

$$W(z) = \int_0^\infty \frac{e^{-u}}{K_1^2(u) + \pi^2 I_1^2(u)} \frac{du}{u} \cdots .$$

Also replace the sentence, "Many *of these* approximations have appeared previously in *MTAC*, v. 7 and v. 8" by "Many *such* approximations have appeared in *MTAC*, v. 7 and v. 8."

CLASSIFICATION OF TABLES

A. Arithmetical Tables. Mathematical Constants

B. Powers

C. Logarithms

D. Circular Functions

E. Hyperbolic and Exponential Functions

F. Theory of Numbers

G. Higher Algebra

H. Numerical Solution of Equations

I. Finite Differences. Interpolation

J. Summation of Series

K. Statistics

L. Higher Mathematical Functions

M. Integrals

N. Interest and Investment

O. Actuarial Science

P. Engineering

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R. Geodesy

S. Physics, Geophysics, Crystallography

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HASTINGS, HAYWARD, & WONG.

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